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Principle of Automatic Control (2)
自动控制原理2 (全英语教学课程)

Topic 3
Nonlinear Control Systems new

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Learning Outcomes for Topic 3

After completing this topic, you will be able to:

- Understand common nonlinear control phenomena
- Apply the most powerful nonlinear analysis methods
- Use some practical nonlinear control design methods

Outline

- Nonlinear models and phenomena
- Computer simulation
- Linearization
- Describing function
- Summary

New terminologies in this topic

Linear Systems

Definition: Let M be a signal space. The system $S : M \rightarrow M$ is linear if for all $u, v \in M$ and $\alpha \in \mathbb{R}$

$$S(\alpha u) = \alpha S(u) \qquad \text{scaling}$$

$$S(u+v) = S(u) + S(v) \qquad \text{superposition}$$

Example: Linear time-invariant systems

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad x(0) = 0$$

$$y(t) = g(t) \star u(t) = \int_0^t g(\tau)u(t-\tau)d\tau$$

$$Y(s) = G(s)U(s)$$

Notice the importance to have zero initial conditions

Linear Systems Have Nice Properties

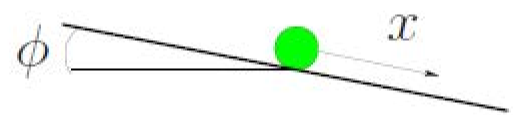
Local stability=global stability Stability if all eigenvalues of A (or poles of $G(s)$) are in the left half-plane

Superposition Enough to know a step (or impulse) response

Frequency analysis possible Sinusoidal inputs give sinusoidal outputs: $Y(i\omega) = G(i\omega)U(i\omega)$

Linear Models may be too Crude Approximations

Example: Positioning of a ball on a beam



Nonlinear model: $m\ddot{x}(t) = mg\sin\phi(t)$, Linear model: $\ddot{x}(t) = g\phi(t)$

Can the ball move 0.1 meter in 0.1 seconds from steady state?

Linear model (step response with $\phi = \phi_0$) gives

$$x(t) \approx 10\frac{t^2}{2}\phi_0 \approx 0.05\phi_0$$

so that

$$\phi_0 \approx \frac{0.1}{0.05} = 2 \text{ rad} = 114^\circ$$

Unrealistic answer. Clearly outside linear region!

Linear model valid only if $\sin\phi \approx \phi$

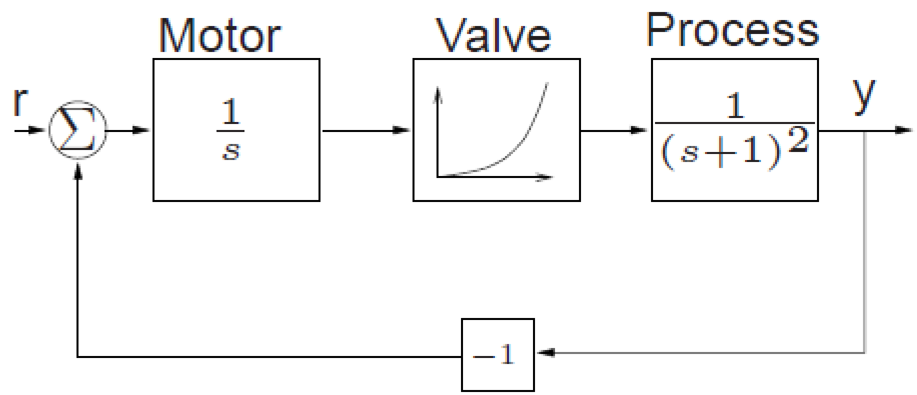
Must consider nonlinear model. Possibly also include other nonlinearities such as centripetal force, saturation, friction etc.

Linear Models are not Rich Enough

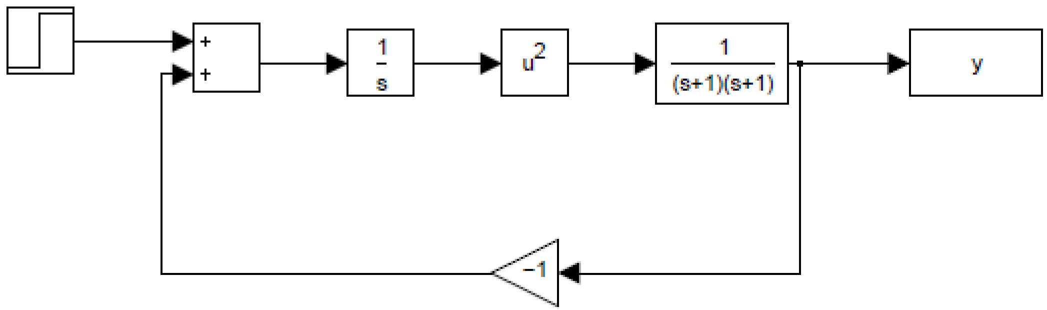
Linear models can not describe many phenomena seen in nonlinear systems

Stability Can Depend on Reference Signal

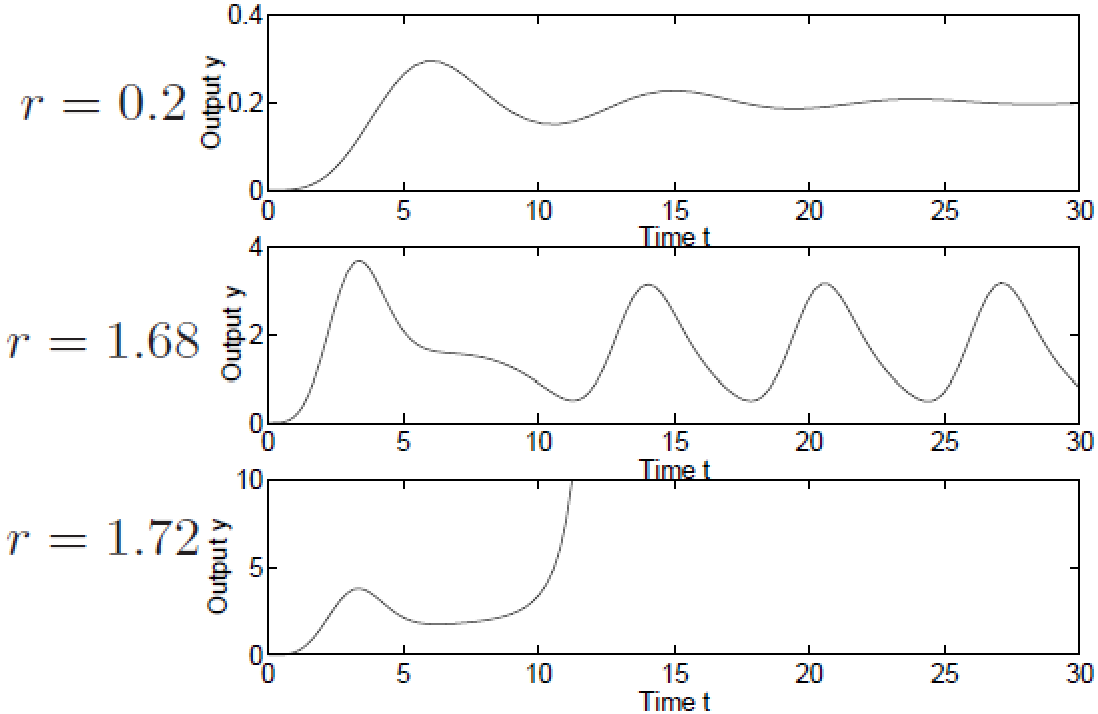
Example: Control system with valve characteristic $f(u) = u^2$



Simulink block diagram:



Step Responses



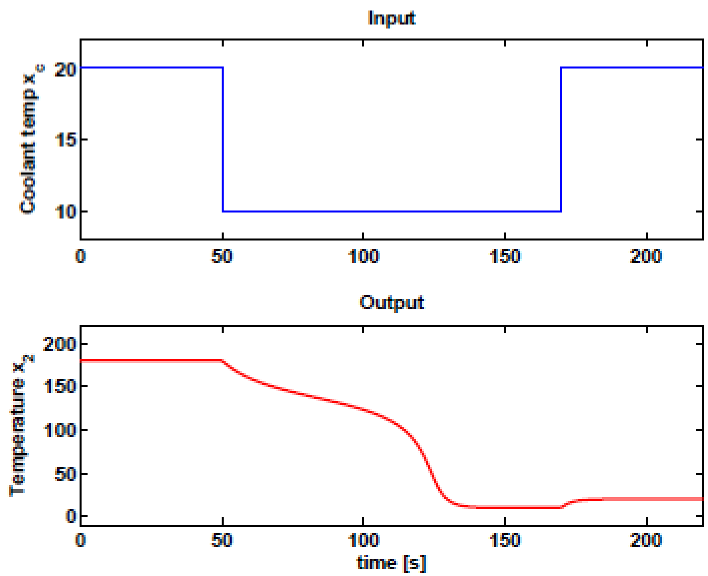
Stability depends on amplitude of the reference signal!

(The linearized gain of the valve increases with increasing amplitude)

Multiple Equilibria

Example: chemical reactor

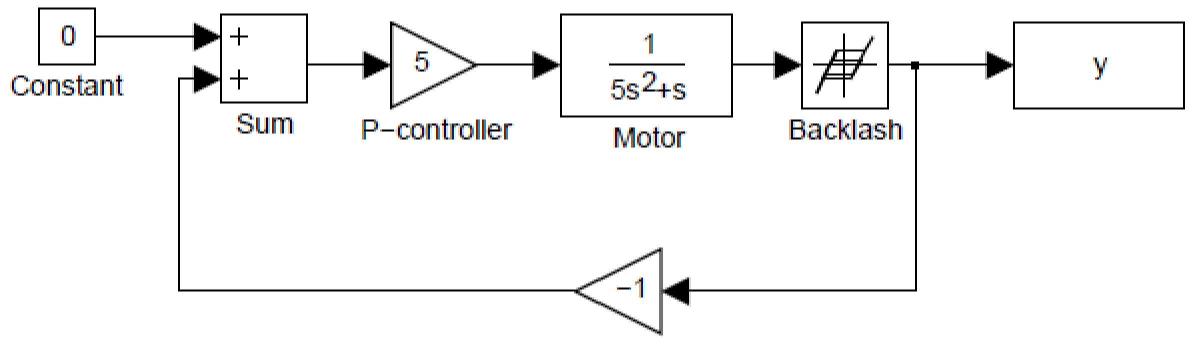
$$\begin{aligned}\dot{x}_1 &= -x_1 exp\left(-\frac{1}{x_2}\right) + f(1 - x_1)\\ \dot{x}_2 &= x_1 exp\left(-\frac{1}{x_2}\right) - \epsilon f(x_2 - x_c)\\ f &= 0.7, \epsilon = 0.4\end{aligned}$$



Existence of multiple stable equilibria for the same input gives hysteresis effect

Stable Periodic Solutions

Example: Position control of motor with back-lash

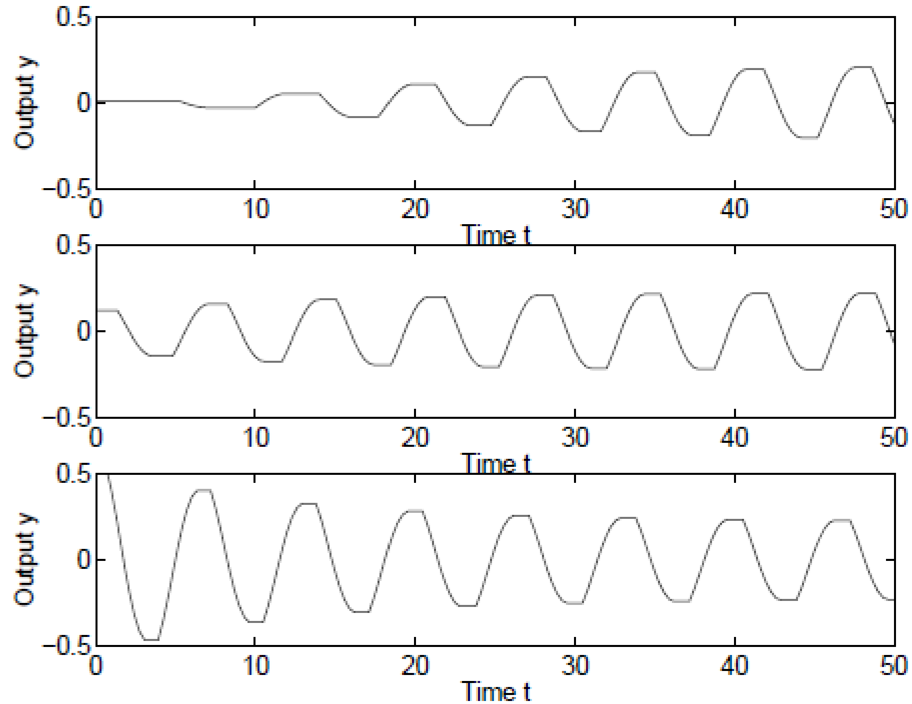


Motor: $G(s) = \frac{1}{s(1+5s)}$

Controller: $K = 5$

Back-lash induces an oscillation

Frequency and amplitude independent of initial conditions:



How predict and avoid oscillations?

Nonlinear Differential Equations

Definition: A solution to

$$\dot{x}(t) = f(x(t)), \quad x(0) = x_0 \tag{1}$$

over an interval $[0, T]$ is a $\mathbf{C}^1$ function $x : [0, T] \to \mathbb{R}^n$ such that (1) is fulfilled.

- When does there exists a solution?
- When is the solution unique?

Example: $\dot{x} = Ax$, $x(0) = x_0$, gives $x(t) = \exp(At)x_0$

Existence Problems

Example: The differential equation $\dot{x} = x^2$, $x(0) = x_0$

has solution $x(t) = \frac{x_0}{1 - x_0 t}$, $\quad 0 \leq t < \frac{1}{x_0}$

Solution not defined for $t_f = \frac{1}{x_0}$

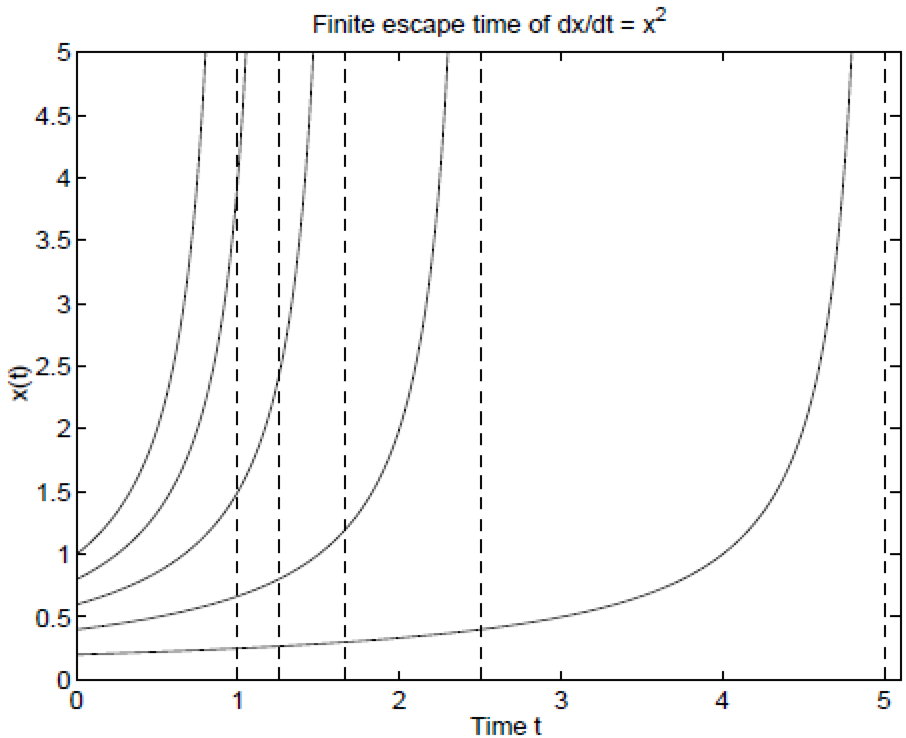
Solution interval depends on initial condition!

Recall the trick: $\dot{x} = x^2 \;\Rightarrow\; \frac{dx}{x^2} = dt$

Integrate $\Rightarrow \frac{-1}{x(t)} - \frac{-1}{x(0)} = t \;\Rightarrow x(t) = \frac{x_0}{1 - x_0 t}$

Finite Escape Time

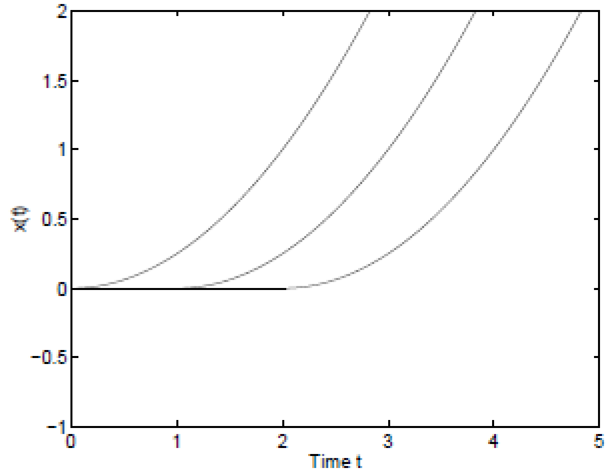
Simulation for various initial conditions x_0



Uniqueness Problems

Example: $\dot{x} = \sqrt{x}$, $x(0) = 0$, has many solutions:

$$x(t) = \begin{cases} (t-C)^2/4 & t > C \\ 0 & t \le C \end{cases}$$



Physical Interpretation

Consider the reverse example, i.e., the water tank lab process with

$$\dot{x} = -\sqrt{x}, \qquad x(T) = 0$$

where x is the water level. It is then impossible to know at what time $t < T$ the level was $x(t) = x_0 > 0$.

Hint: Reverse time $s = T - t \Rightarrow ds = -dt$ and thus

$$\frac{dx}{ds} = -\frac{dx}{dt}$$

State-Space Models

State x, input u, output y

General: $f(x, u, y, \dot{x}, \dot{u}, \dot{y}, \ldots) = 0$

Explicit: $\dot{x} = f(x, u), \quad y = h(x)$

Affine in u: $\dot{x} = f(x) + g(x)u, \quad y = h(x)$

Linear: $\dot{x} = Ax + Bu, \quad y = Cx$

Transformation to Autonomous System

A nonautonomous system

$$\dot{x} = f(x, t)$$

is always possible to transform to an autonomous system by introducing $x_{n+1} = t$:

$$\dot{x} = f(x, x_{n+1})$$
$$\dot{x}_{n+1} = 1$$

Transformation to First-Order System

Given a differential equation in y with highest derivative $\frac{d^n y}{dt^n}$,

express the equation in $x = \begin{pmatrix} y & \frac{dy}{dt} & \dots & \frac{d^{n-1}y}{dt^{n-1}} \end{pmatrix}^T$ **Example**:

Pendulum

$$MR^2\ddot{\theta} + k\dot{\theta} + MgR\sin\theta = 0$$

$x = \begin{pmatrix} \theta & \dot{\theta} \end{pmatrix}^T$ gives

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{k}{MR^2}x_2 - \frac{g}{R}\sin x_1$$

Equilibria

Definition: A point (x^*, u^*, y^*) is an equilibrium, if a solution starting in (x^*, u^*, y^*) stays there forever.

Corresponds to putting all derivatives to zero:

General:	$f(x^*, u^*, y^*, 0, 0, \ldots) = 0$
Explicit:	$0 = f(x^*, u^*), \quad y^* = h(x^*)$
Affine in u:	$0 = f(x^*) + g(x^*)u^*, \quad y^* = h(x^*)$
Linear:	$0 = Ax^* + Bu^*, \quad y^* = Cx^*$

Often the equilibrium is defined only through the state x^*

Multiple Equilibria

Example: Pendulum

$$MR^2\ddot{\theta} + k\dot{\theta} + MgR\sin\theta = 0$$

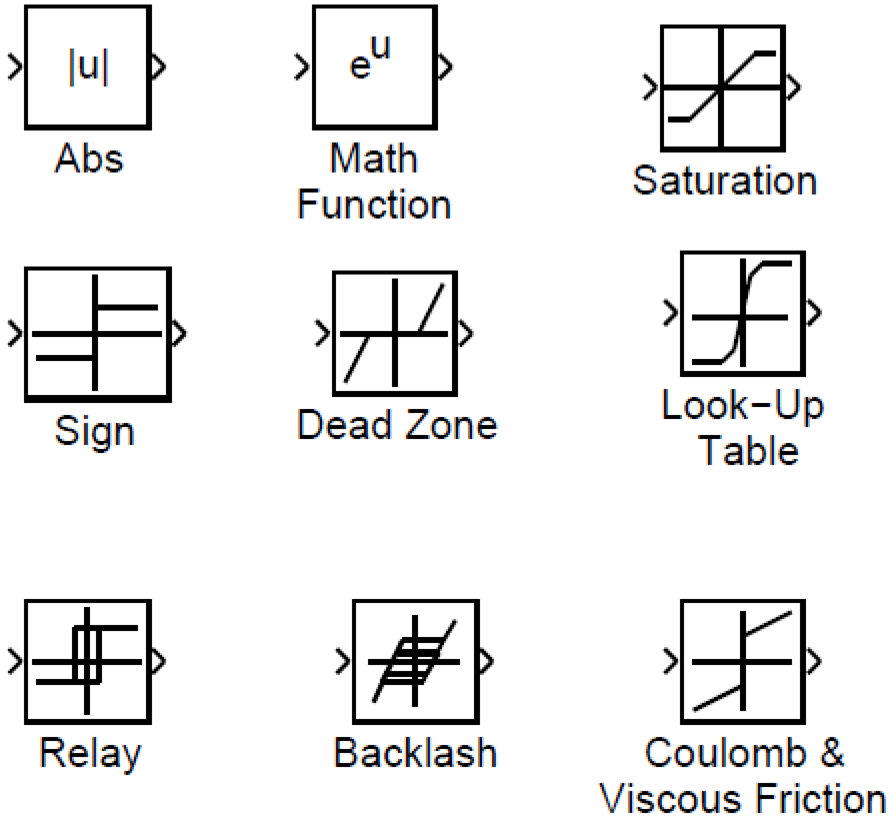
$\ddot{\theta} = \dot{\theta} = 0$ gives $\sin\theta = 0$ and thus $\theta^* = k\pi$

Alternatively in first-order form:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{k}{MR^2}x_2 - \frac{g}{R}\sin x_1$$

$\dot{x}_1 = \dot{x}_2 = 0$ gives $x_2^* = 0$ and $\sin(x_1^*) = 0$

Some Common Nonlinearities in Control Systems



When do we need Nonlinear Analysis & Design?

- When the system is strongly nonlinear
- When the range of operation is large
- When distinctive nonlinear phenomena are relevant
- When we want to push performance to the limit

Analysis Through Simulation

Simulation tools:

ODE's $\dot{x} = f(t, x, u)$

- ACSL, Simnon, Simulink

DAE's $F(t, \dot{x}, x, u) = 0$

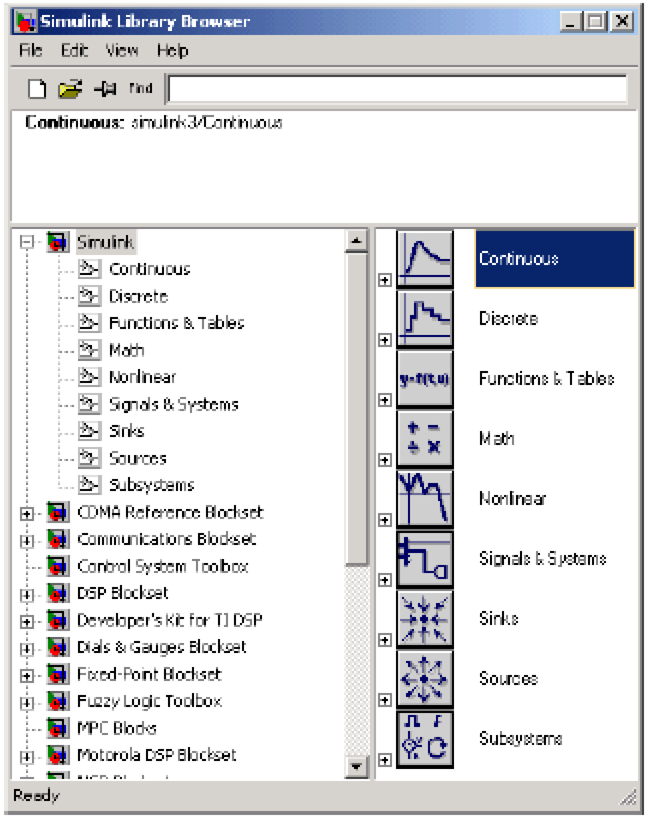
- Omsim, Dymola, Modelica
  `http://www.modelica.org`

Special purpose simulation tools

- Spice, EMTP, ADAMS, gPROMS

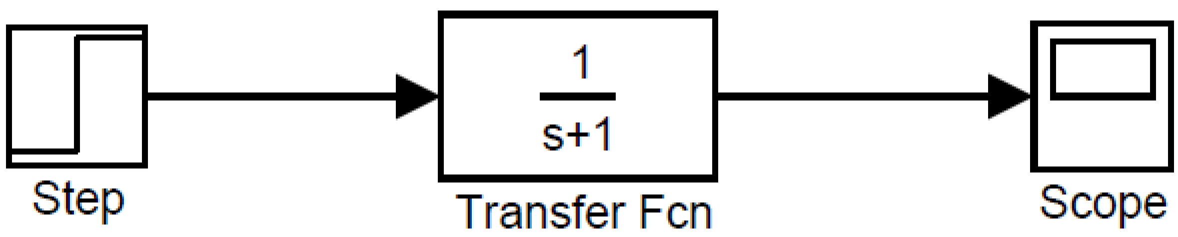
Simulink

```
> matlab
>> simulink
```

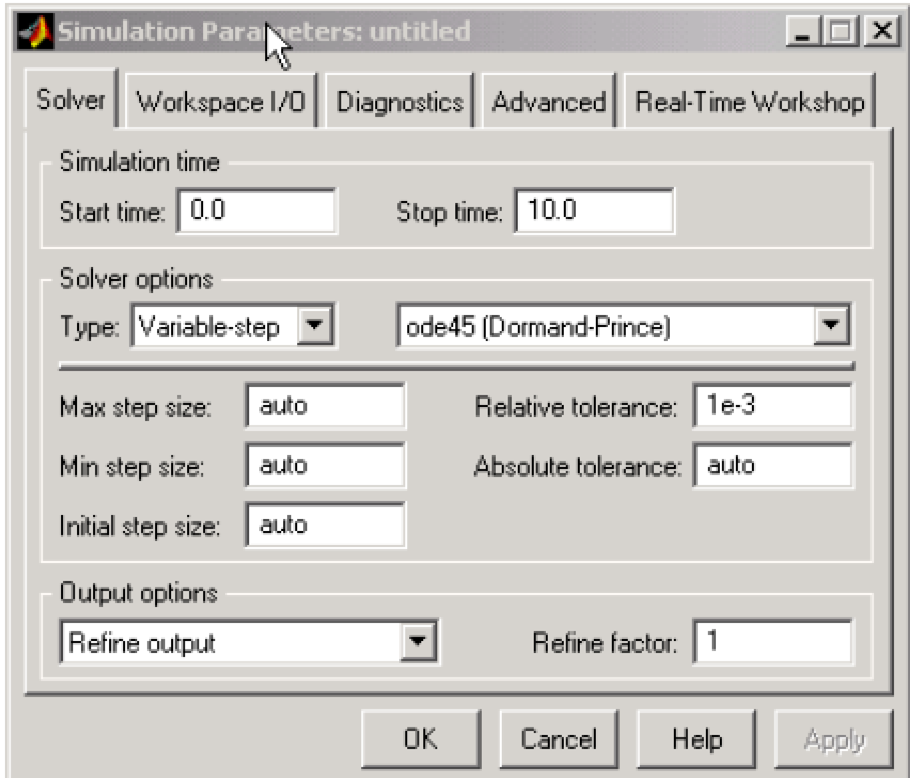


An Example in Simulink

```
File -> New -> Model
Double click on Continous
Transfer Fcn
Step (in Sources)
Scope (in Sinks)
Connect (mouse-left)
Simulation -> Parameters
```



Choose Simulation Parameters



Don’t forget “Apply”

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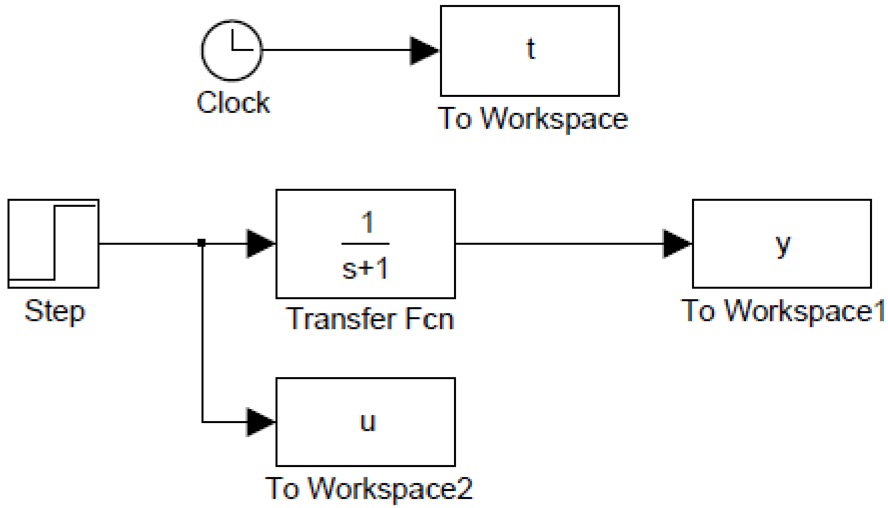
Save Results to Workspace

stepmodel.mdl



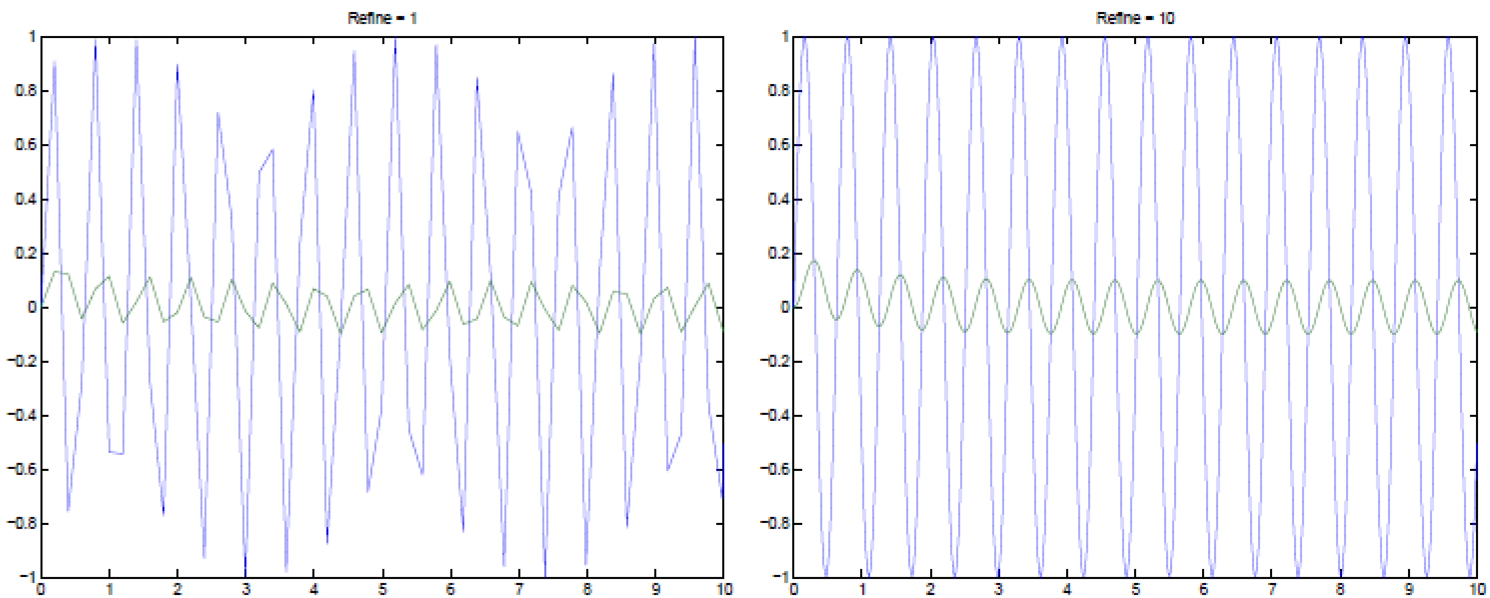
Check “Save format” of output blocks (“Array” instead of “Structure”)

```
>> plot(t,y)
```

How To Get Better Accuracy

Modify Refine, Absolute and Relative Tolerances, Integration method

Refine adds interpolation points:



Use Scripts to Document Simulations

If the block-diagram is saved to `stepmodel.mdl`,
the following Script-file `simstepmodel.m` simulates the system:

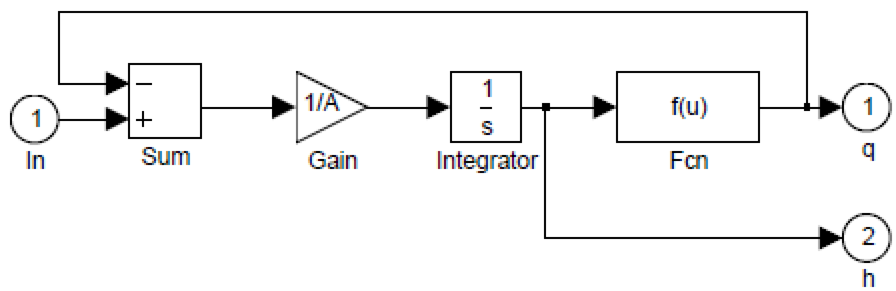
```
open_system('stepmodel')
set_param('stepmodel','RelTol','1e-3')
set_param('stepmodel','AbsTol','1e-6')
set_param('stepmodel','Refine','1')
tic
sim('stepmodel',6)
toc
subplot(2,1,1),plot(t,y),title('y')
subplot(2,1,2),plot(t,u),title('u')
```

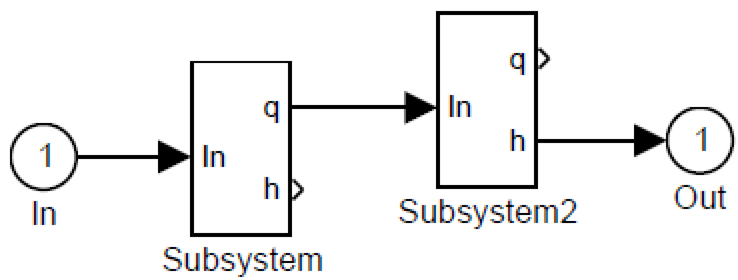
Example: Two-Tank System

The system consists of two identical tank models:

$$\dot{h} = (u - q)/A$$

$$q = a\sqrt{2g}\sqrt{h}$$





Linearization in Simulink

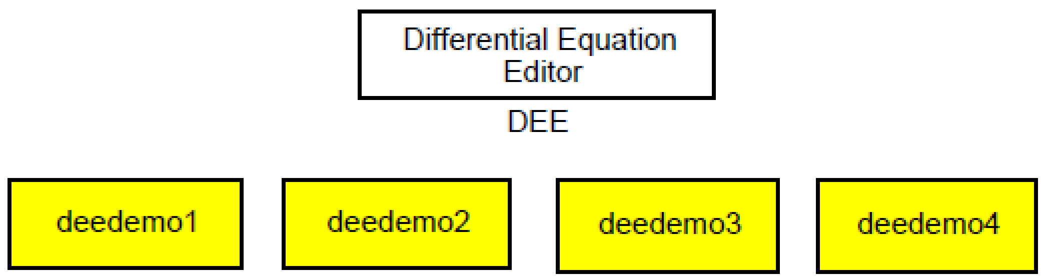
Linearize about equilibrium (x_0, u_0, y_0):

```
>> A=2.7e-3;a=7e-6,g=9.8;
>> [x0,u0,y0]=trim('twotank',[0.1 0.1]',[],0.1)
x0 =
     0.1000
     0.1000
u0 =
    9.7995e-006
y0 =
     0.1000
>> [aa,bb,cc,dd]=linmod('twotank',x0,u0);
>> sys=ss(aa,bb,cc,dd);
>> bode(sys)
```

Differential Equation Editor

dee is a Simulink-based differential equation editor

```
>> dee
```



Run the demonstrations

Phase-Plane Analysis

- Download ICTools from

  `http://www.control.lth.se/~ictools`

- Down load DFIELD and PPLANE from

  `http://math.rice.edu/~dfield`

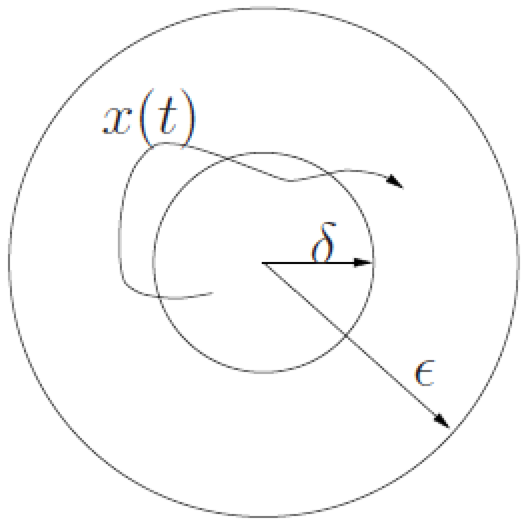
  This was the preferred tool last year!

Local Stability

Consider $\dot{x} = f(x)$ with $f(0) = 0$

Definition: The equilibrium $x^* = 0$ is **stable** if for all $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that

$$\|x(0)\| < \delta \quad \Rightarrow \quad \|x(t)\| < \epsilon, \quad \forall t \geq 0$$



If $x^* = 0$ is not stable it is called **unstable**.

Asymptotic Stability

Definition: The equilibrium $x = 0$ is **asymptotically stable** if it is stable and δ can be chosen such that

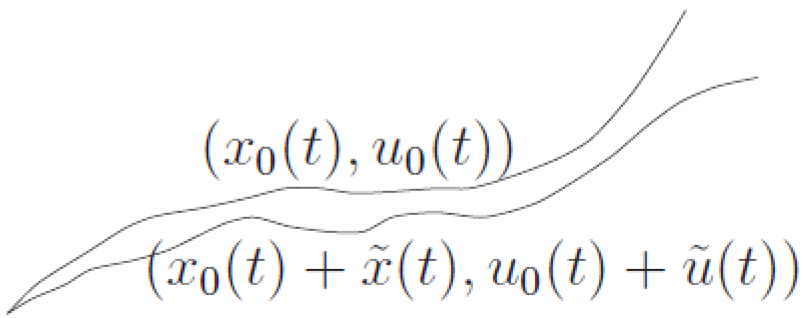
$$\|x(0)\| < \delta \quad \Rightarrow \quad \lim_{t \to \infty} x(t) = 0$$

The equilibrium is **globally asymptotically stable** if it is stable and $\lim_{t \to \infty} x(t) = 0$ for all $x(0)$.

Linearization Around a Trajectory

Let $(x_0(t), u_0(t))$ denote a solution to $\dot{x} = f(x,u)$ and consider another solution $(x(t), u(t)) = (x_0(t) + \tilde{x}(t), u_0(t) + \tilde{u}(t))$:

$$\begin{aligned}\dot{x}(t) &= f(x_0(t) + \tilde{x}(t), u_0(t) + \tilde{u}(t)) \\ &= f(x_0(t), u_0(t)) + \frac{\partial f}{\partial x}(x_0(t), u_0(t))\tilde{x}(t) \\ &\quad + \frac{\partial f}{\partial u}(x_0(t), u_0(t))\tilde{u}(t) + \mathcal{O}(\|\tilde{x}, \tilde{u}\|^2)\end{aligned}$$



Hence, for small $(\tilde{x}, \tilde{u})$, approximately

$$\dot{\tilde{x}}(t) = A(x_0(t), u_0(t))\tilde{x}(t) + B(x_0(t), u_0(t))\tilde{u}(t)$$

where

$$A(x_0(t), u_0(t)) = \frac{\partial f}{\partial x}(x_0(t), u_0(t))$$
$$B(x_0(t), u_0(t)) = \frac{\partial f}{\partial u}(x_0(t), u_0(t))$$

Note that A and B are time dependent. However, if $(x_0(t), u_0(t)) \equiv (x_0, u_0)$ then A and B are constant.

Pointwise Left Half-Plane Eigenvalues of A(t) Do *Not Impose Stability*

$$A(t) = \begin{pmatrix} -1 + \alpha \cos^2 t & 1 - \alpha \sin t \cos t \\ -1 - \alpha \sin t \cos t & -1 + \alpha \sin^2 t \end{pmatrix}, \quad \alpha > 0$$

Pointwise eigenvalues are given by

$$\lambda(t) = \lambda = \frac{\alpha - 2 \pm \sqrt{\alpha^2 - 4}}{2}$$

which are stable for $0 < \alpha < 2$. However,

$$x(t) = \begin{pmatrix} e^{(\alpha-1)t} \cos t & e^{-t} \sin t \\ -e^{(\alpha-1)t} \sin t & e^{-t} \cos t \end{pmatrix} x(0),$$

is unbounded solution for $\alpha > 1$.

Lyapunov's Linearization Method

Theorem: Let x_0 be an equilibrium of $\dot{x} = f(x)$ with $f \in \mathbb{C}^1$. Denote $A = \frac{\partial f}{\partial x}(x_0)$ and $\alpha(A) = \max \mathrm{Re}(\lambda(A))$.

- If $\alpha(A) < 0$, then x_0 is asymptotically stable
- If $\alpha(A) > 0$, then x_0 is unstable

The fundamental result for linear systems theory!

The case $\alpha(A) = 0$ needs further investigation.

The theorem is also called *Lyapunov's Indirect Method*.

A proof is given next lecture.

Example

The linearization of

$$\dot{x}_1 = -x_1^2 + x_1 + \sin x_2$$
$$\dot{x}_2 = \cos x_2 - x_1^3 - 5x_2$$

at the equilibrium $x_0 = (1, 0)^T$ is given by

$$A = \begin{pmatrix} -1 & 1 \\ -3 & -5 \end{pmatrix}, \qquad \lambda(A) = \{-2, -4\}$$

x_0 is thus an asymptotically stable equilibrium for the *nonlinear* system.

Linear Systems Revival

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Analytic solution: $x(t) = e^{At}x(0), t \geq 0.$

If A is diagonalizable, then

$$e^{At} = Ve^{\Lambda t}V^{-1} = \begin{bmatrix} v_1 & v_2 \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{-1}$$

where v_1, v_2 are the eigenvectors of A ($Av_1 = \lambda_1 v_1$ etc).

This implies that

$$x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2,$$

where the constants c_1 and c_2 are given by the initial conditions

Example: Two real negative eigenvalues

Given the eigenvalues $\lambda_1 < \lambda_2 < 0$, with corresponding eigenvectors v_1 and v_2, respectively.

Solution: $x(t) = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$

Slow eigenvalue/vector: $x(t) \approx c_2 e^{\lambda_2 t} v_2$ for large t.
Moves along the slow eigenvector towards $x = 0$ for large t

Fast eigenvalue/vector: $x(t) \approx c_1 e^{\lambda_1 t} v_1 + c_2 v_2$ for small t.
Moves along the fast eigenvector for small t

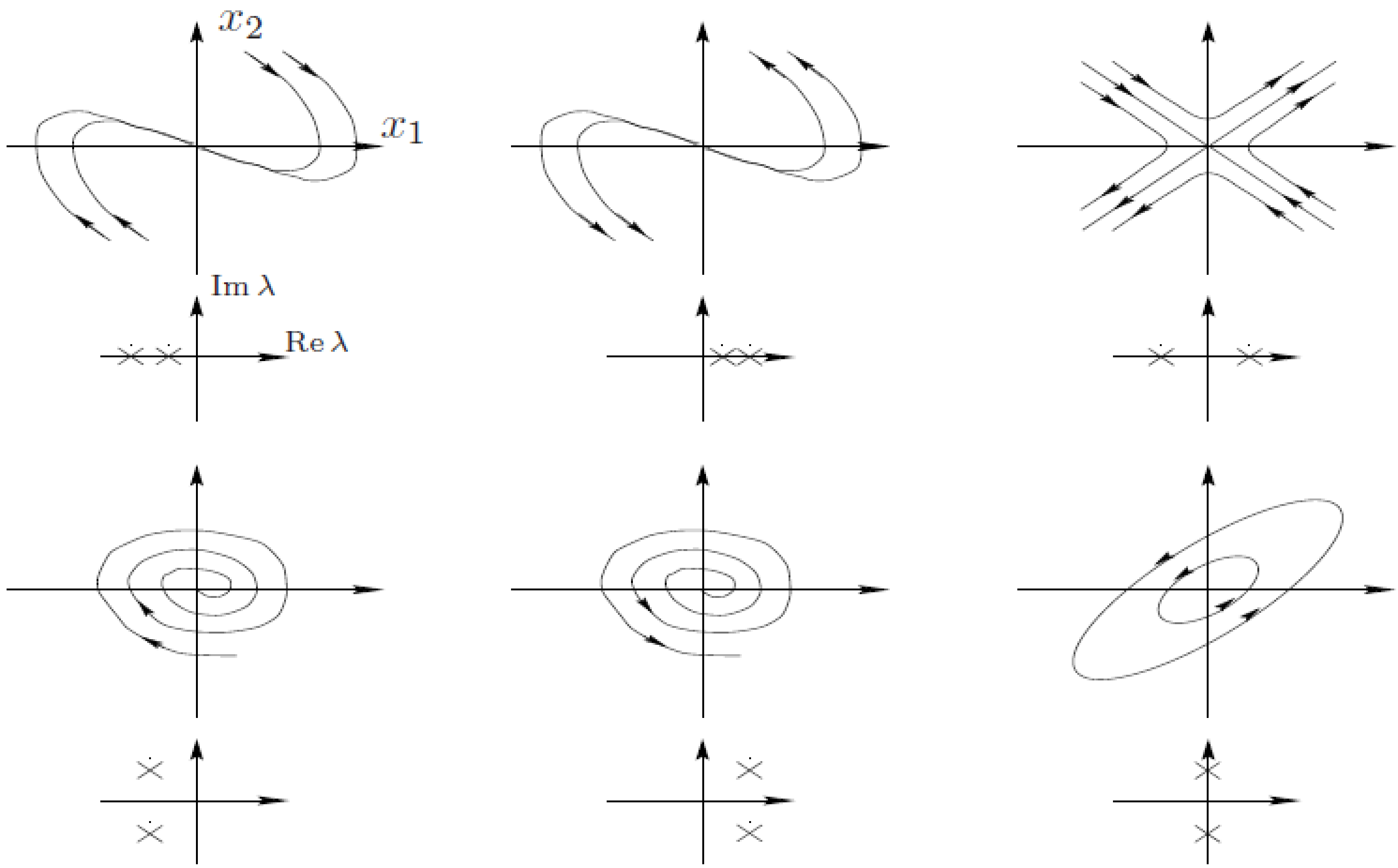
Phase-Plane Analysis for Linear Systems

The location of the eigenvalues $\lambda(A)$ determines the characteristics of the trajectories.

Six cases:

	stable node	unstable node	saddle point
$\operatorname{Im} \lambda_i = 0:$	$\lambda_1, \lambda_2 < 0$	$\lambda_1, \lambda_2 > 0$	$\lambda_1 < 0 < \lambda_2$
$\operatorname{Im} \lambda_i \neq 0:$	$\operatorname{Re} \lambda_i < 0$	$\operatorname{Re} \lambda_i > 0$	$\operatorname{Re} \lambda_i = 0$
	stable focus	unstable focus	center point

Equilibrium Points for Linear Systems



Example—Unstable Focus

$$\dot{x} = \begin{bmatrix} \sigma & -\omega \\ \omega & \sigma \end{bmatrix} x, \qquad \sigma, \omega > 0, \qquad \lambda_{1,2} = \sigma \pm i\omega$$

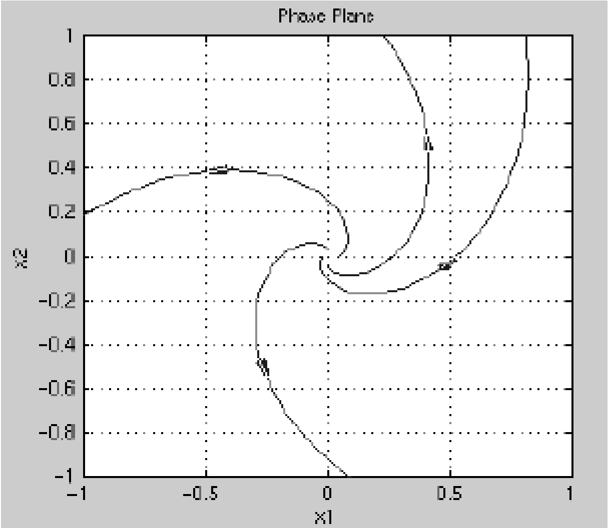
$$x(t) = e^{At}x(0) = \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix} \begin{bmatrix} e^{\sigma t}e^{i\omega t} & 0 \\ 0 & e^{\sigma t}e^{-i\omega t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}^{-1} x(0)$$

In polar coordinates $r = \sqrt{x_1^2 + x_2^2}, \theta = \arctan x_2/x_1$
$(x_1 = r\cos\theta, x_2 = r\sin\theta)$:

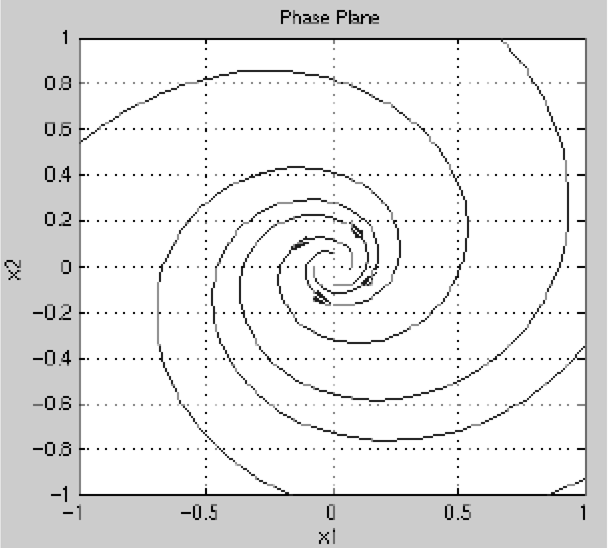
$$\dot{r} = \sigma r$$

$$\dot{\theta} = \omega$$

$$\lambda_{1,2} = 1 \pm i$$



$$\lambda_{1,2} = 0.3 \pm i$$

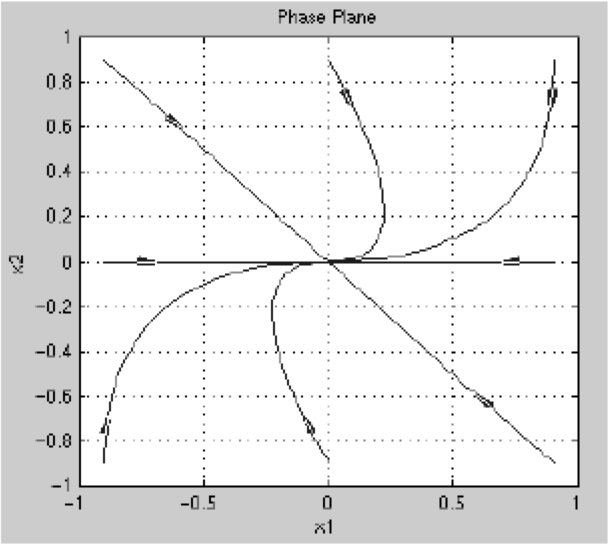


Example—Stable Node

$$\dot{x} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix} x$$

$(\lambda_1, \lambda_2) = (-1, -2)$ and $\begin{bmatrix} v_1 & v_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

v_1 is the slow direction and v_2 is the fast.



Fast: $x_2 = -x_1 + c_3$
Slow: $x_2 = 0$

Phase-Plane Analysis for Nonlinear Systems

Close to equilibrium points "nonlinear system" $\approx$ "linear system"

Theorem: Assume

$$\dot{x} = f(x) = Ax + g(x),$$

with $\lim_{\|x\|\to 0} \|g(x)\|/\|x\| = 0$. If $\dot{z} = Az$ has a focus, node, or saddle point, then $\dot{x} = f(x)$ has the same type of equilibrium at the origin.

Remark: If the linearized system has a center, then the nonlinear system has either a center or a focus.

How to Draw Phase Portraits

By hand:

1. Find equilibria
2. Sketch local behavior around equilibria
3. Sketch $(\dot{x}_1, \dot{x}_2)$ for some other points. Notice that

$$\frac{dx_2}{dx_1} = \frac{\dot{x}_1}{\dot{x}_2}$$

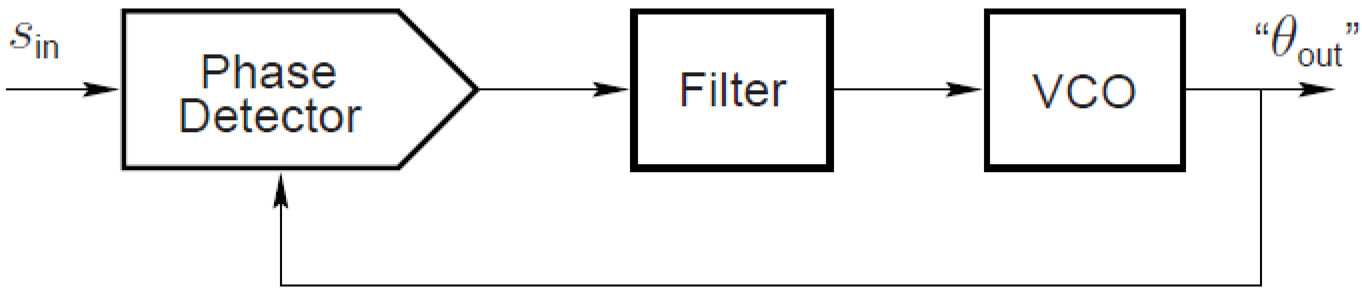
4. Try to find possible periodic orbits
5. Guess solutions

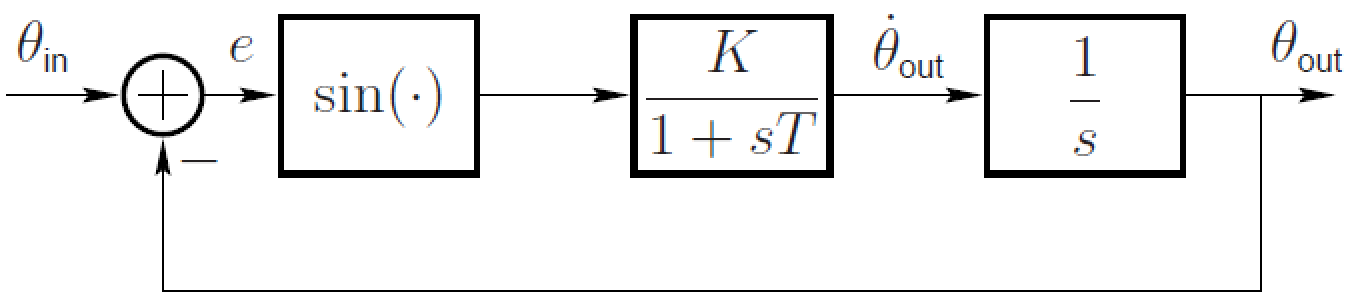
By computer:

1. Matlab: dee or pplane

Phase-Locked Loop

A PLL tracks phase $\theta_{\text{in}}(t)$ of a signal $s_{\text{in}}(t) = A\sin[\omega t + \theta_{\text{in}}(t)]$.





Phase-Plane Analysis of PLL

Let $(x_1, x_2) = (\theta_{\text{out}}, \dot{\theta}_{\text{out}})$, $K, T > 0$, and $\theta_{\text{in}}(t) \equiv \theta_{\text{in}}$.

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = -T^{-1}x_2(t) + KT^{-1}\sin(\theta_{\text{in}} - x_1(t))$$

Equilibria are $(\theta_{\text{in}} + n\pi, 0)$ since

$$\dot{x}_1 = 0 \Rightarrow x_2 = 0$$
$$\dot{x}_2 = 0 \Rightarrow \sin(\theta_{in} - x_1) = 0 \Rightarrow x_1 = \theta_{\text{in}} + n\pi$$

Classification of Equilibria

Linearization gives the following characteristic equations:

n even:

$$\lambda^2 + T^{-1}\lambda + KT^{-1} = 0$$

$K > (4T)^{-1}$ gives stable focus
$0 < K < (4T)^{-1}$ gives stable node

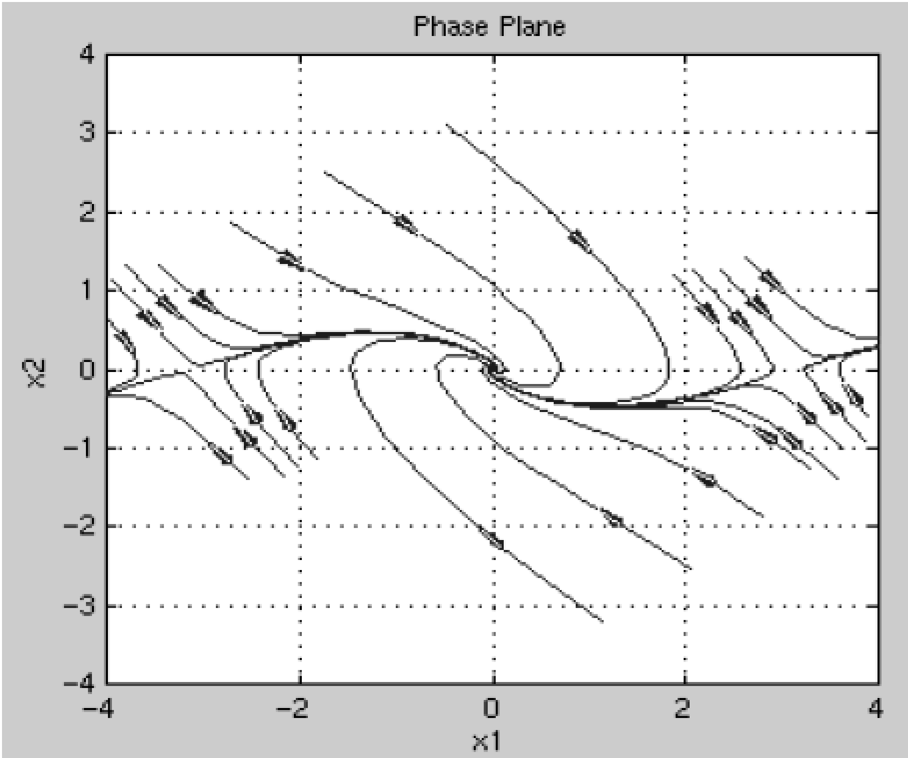
n odd:

$$\lambda^2 + T^{-1}\lambda - KT^{-1} = 0$$

Saddle points for all $K, T > 0$

Phase-Plane for PLL

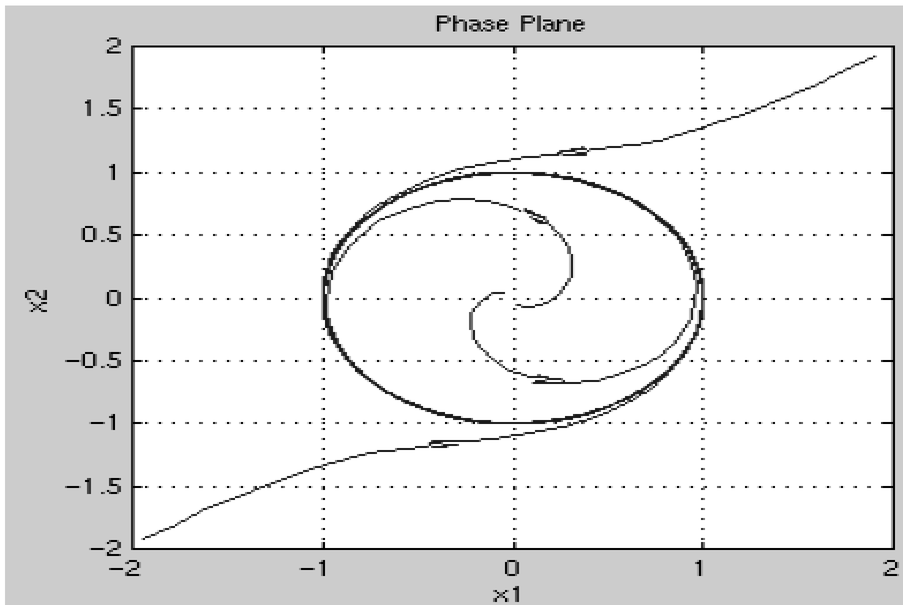
$(K,T)=(1/2,1)$: focuses $(2k\pi,0)$, saddle points $((2k+1)\pi,0)$



Periodic Solutions

Example of an asymptotically stable periodic solution:

$$\dot{x}_1 = x_1 - x_2 - x_1(x_1^2 + x_2^2)$$
$$\dot{x}_2 = x_1 + x_2 - x_2(x_1^2 + x_2^2) \qquad (1)$$



Periodic solution: Polar coordinates.

$$x_1 = r\cos\theta \quad \Rightarrow \quad \dot{x}_1 = \cos\theta\dot{r} - r\sin\theta\dot{\theta}$$

$$x_2 = r\sin\theta \quad \Rightarrow \quad \dot{x}_2 = \sin\theta\dot{r} + r\cos\theta\dot{\theta}$$

implies

$$\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \frac{1}{r}\begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix}$$

Now

$$\dot{x}_1 = r(1-r^2)\cos\theta - r\sin\theta$$

$$\dot{x}_2 = r(1-r^2)\sin\theta + r\cos\theta$$

gives

$$\dot{r} = r(1-r^2)$$

$$\dot{\theta} = 1$$

Only $r = 1$ is a stable equilibrium!

A system has a **periodic solution** if for some $T > 0$

$$x(t+T) = x(t), \qquad \forall t \geq 0$$

A **periodic orbit** is the image of x in the phase portrait.

- When does there exist a periodic solution?
- When is it stable?

Note that $x(t) \equiv$ const is by convention not regarded periodic

Flow

The solution of $\dot{x} = f(x)$ is sometimes denoted

$$\phi_t(x_0)$$

to emphasize the dependence on the initial point $x_0 \in \mathbb{R}^n$

$\phi_t(\cdot)$ is called the **flow**.

Poincaré Map

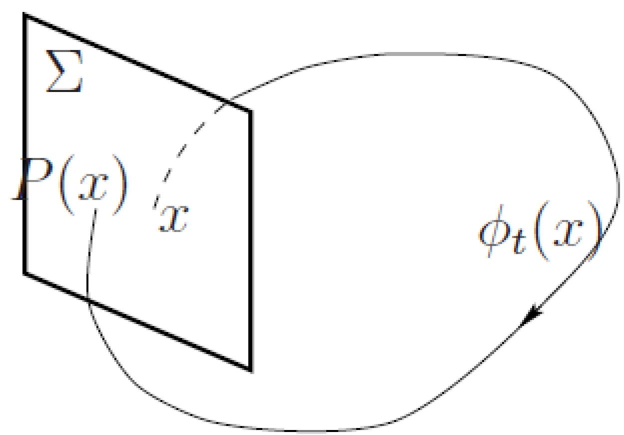
Assume $\phi_t(x_0)$ is a periodic solution with period T.

Let $\Sigma \subset \mathbb{R}^n$ be an $n-1$-dim hyperplane transverse to f at x_0.

Definition: The Poincaré map $P : \Sigma \to \Sigma$ is

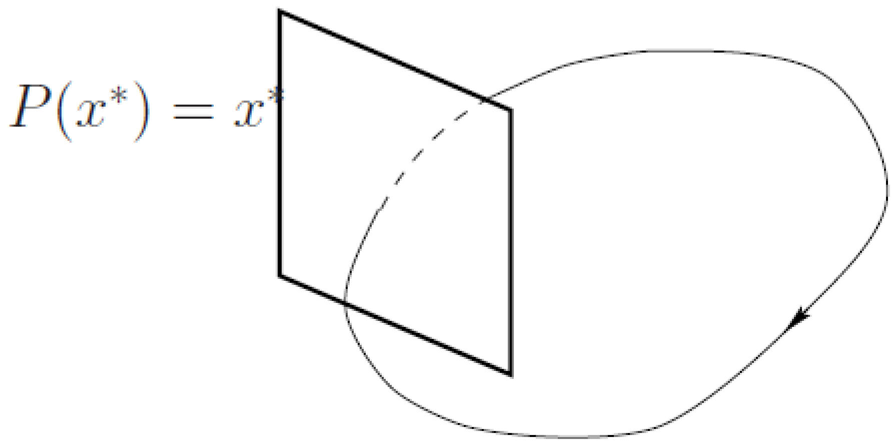
$$P(x) = \phi_{\tau(x)}(x)$$

where $\tau(x)$ is the time of first return.



Existence of Periodic Orbits

A point x^* such that $P(x^*) = x^*$ corresponds to a periodic orbit.



x^* is called a **fixed point** of P.

Stable Periodic Orbit

The linearization of P around x^* gives a matrix W such that

$$P(x) \approx Wx$$

if x is close to x^*.

- $\lambda_j(W) = 1$ for some j
- If $|\lambda_i(W)| < 1$ for all $i \neq j$, then the corresponding periodic orbit is **asymptotically stable**
- If $|\lambda_i(W)| > 1$ for some i, then the periodic orbit is **unstable**.

Example—Stable Unit Circle

Rewrite (1) in polar coordinates:

$$\dot{r} = r(1 - r^2)$$
$$\dot{\theta} = 1$$

Choose $\Sigma = \{(r, \theta) : r > 0, \theta = 2\pi k\}$.

The solution is

$$\phi_t(r_0, \theta_0) = \left([1 + (r_0^{-2} - 1)e^{-2t}]^{-1/2}, t + \theta_0 \right)$$

First return time from any point $(r_0, \theta_0) \in \Sigma$ is

$$\tau(r_0, \theta_0) = 2\pi.$$

The Poincaré map is

$$P(r_0, \theta_0) = \begin{pmatrix} [1 + (r_0^{-2} - 1)e^{-2\cdot 2\pi}]^{-1/2} \\ \theta_0 + 2\pi \end{pmatrix}$$

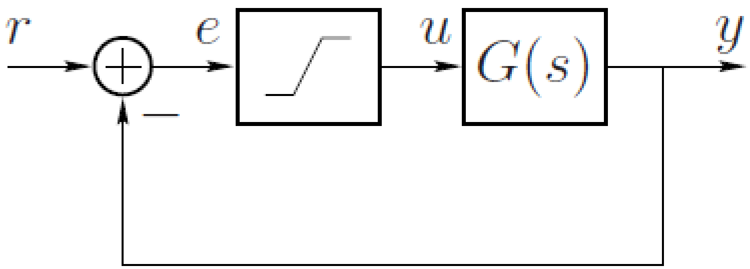
$(r_0, \theta_0) = (1, 2\pi k)$ is a fixed point.

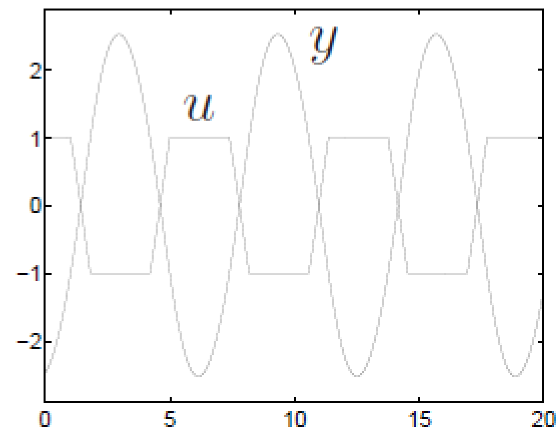
The periodic solution that corresponds to $(r(t), \theta(t)) = (1, t)$ is asymptotically stable because

$$W = \frac{dP}{d(r_0, \theta_0)}(1, 2\pi k) = \begin{pmatrix} e^{-4\pi} & 0 \\ 0 & 1 \end{pmatrix}$$

$\Rightarrow$ Stable periodic orbit (as we already knew for this example) !

Motivating Example





$$G(s) = \frac{4}{s(s+1)^2}$$ and $u = \text{sat}\, e$ give a stable oscillation.

- How can the oscillation be predicted?

A Frequency Response Approach

Nyquist / Bode:

> *A (linear) feedback system will have sustained oscillations (center) if the loop-gain is* 1 *at the frequency where the phase lag is* -180^o

But, can we talk about the frequency response, in terms of gain and phase lag, of a static nonlinearity?

Fourier Series

A periodic function $u(t) = u(t+T)$ has a Fourier series expansion

$$u(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\omega t + b_n \sin n\omega t)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2}\,\sin[n\omega t + \arctan(a_n/b_n)]$$

where $\omega = 2\pi/T$ and

$$a_n(\omega) = \frac{2}{T}\int_0^T u(t)\cos n\omega t\, dt, \quad b_n(\omega) = \frac{2}{T}\int_0^T u(t)\sin n\omega t\, dt$$

Note: Sometimes we make the change of variable $t \to \phi/\omega$

The Fourier Coefficients are Optimal

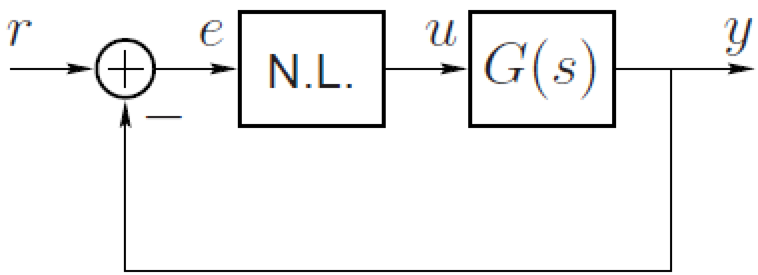
The finite expansion

$$\widehat{u}_k(t) = \frac{a_0}{2} + \sum_{n=1}^{k} (a_n \cos n\omega t + b_n \sin n\omega t)$$

solves

$$\min_{\hat{u}} \frac{2}{T} \int_0^T \left[u(t) - \widehat{u}_k(t) \right]^2 dt$$

The Key Idea



$e(t) = A \sin \omega t$ gives

$$u(t) = \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \sin[n\omega t + \arctan(a_n/b_n)]$$

If $|G(in\omega)| \ll |G(i\omega)|$ for $n \geq 2$, then $n = 1$ suffices, so that

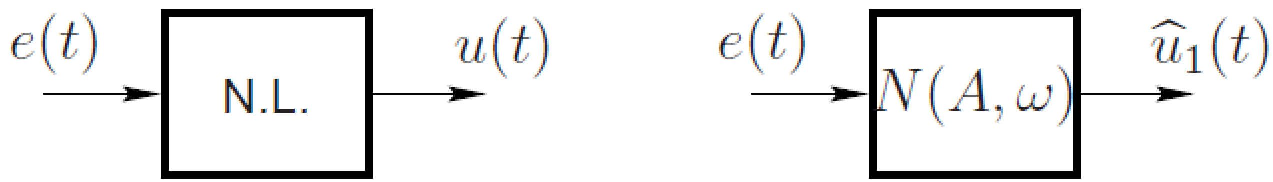
$$y(t) \approx |G(i\omega)| \sqrt{a_1^2 + b_1^2} \sin[\omega t + \arctan(a_1/b_1) + \arg G(i\omega)]$$

That is, we assume all higher harmonics are filtered out by G

Definition of Describing Function

The **describing function** is

$$N(A,\omega) = \frac{b_1(\omega) + i a_1(\omega)}{A}$$



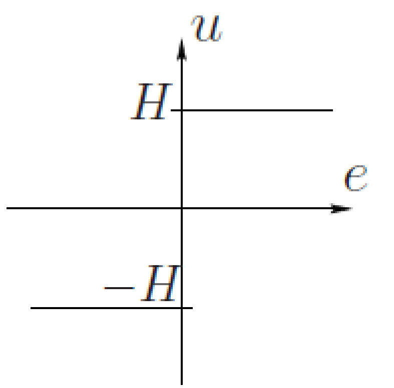
If G is low pass and $a_0 = 0$, then

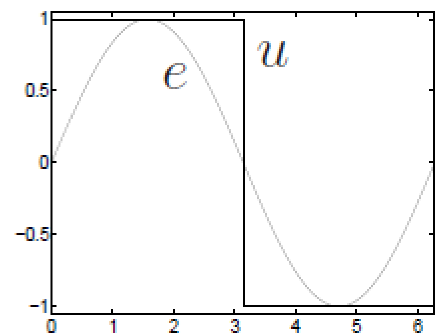
$$\widehat{u}_1(t) = |N(A,\omega)| A \sin[\omega t + \arg N(A,\omega)]$$

can be used instead of $u(t)$ to analyze the system.

Amplitude dependent gain and phase shift!

Describing Function for a Relay





$$\phi = 2\pi t/T$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} u(\phi)\cos\phi\, d\phi = 0$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} u(\phi)\sin\phi\, d\phi = \frac{2}{\pi}\int_0^{\pi} H\sin\phi\, d\phi = \frac{4H}{\pi}$$

The describing function for a relay is thus

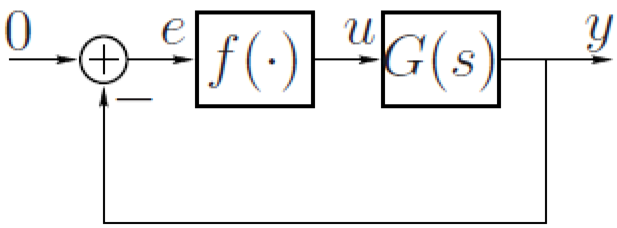
$$N(A) = \frac{b_1(\omega) + i a_1(\omega)}{A} = \frac{4H}{\pi A}$$

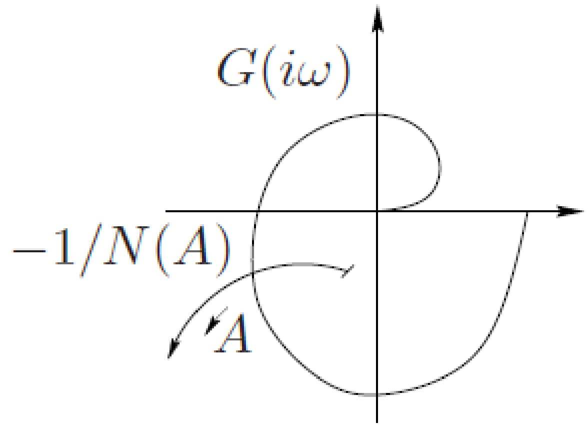
Odd Static Nonlinearities

Assume $f(\cdot)$ and $g(\cdot)$ are odd (i.e. $f(-e) = -f(e)$) static nonlinearities with describing functions N_f and N_g. Then,

- $\operatorname{Im} N_f(A, \omega) = 0$
- $N_f(A, \omega) = N_f(A)$
- $N_{\alpha f}(A) = \alpha N_f(A)$
- $N_{f+g}(A) = N_f(A) + N_g(A)$

Existence of Periodic Solutions



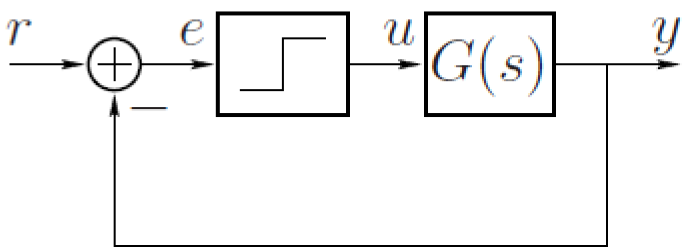


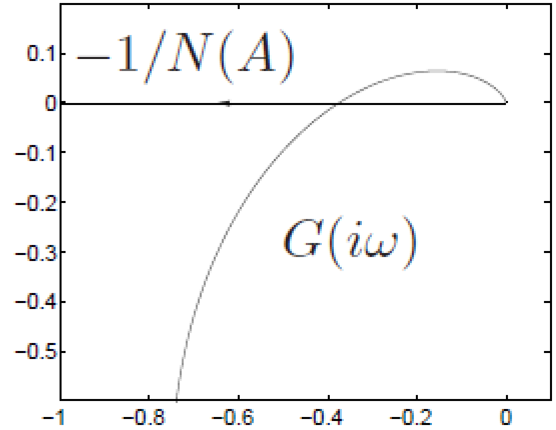
Proposal: sustained oscillations if loop-gain 1 *and phase-lag* -180^o

$$G(i\omega)N(A) = -1 \quad \Leftrightarrow \quad G(i\omega) = -1/N(A)$$

The intersections of the curves $G(i\omega)$ and $-1/N(A)$ give ω and A for a possible periodic solution.

Periodic Solutions in Relay System





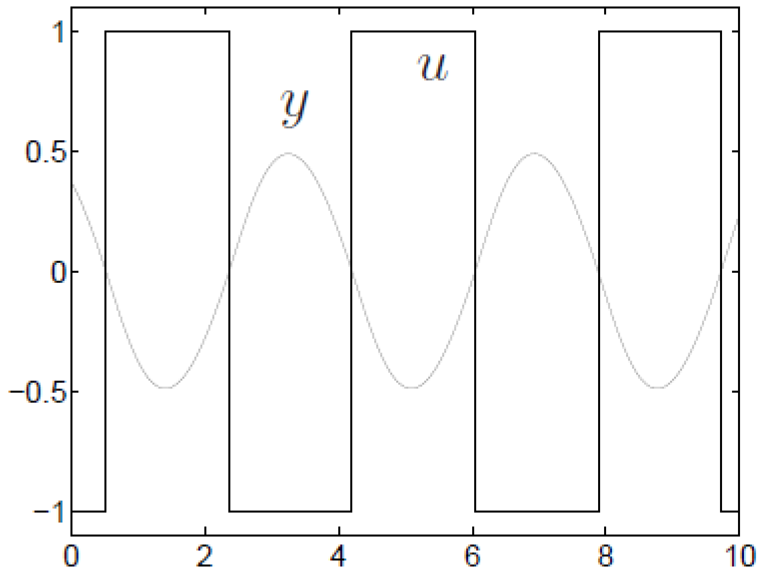
$$G(s) = \frac{3}{(s+1)^3}$$ with feedback $u = -\text{sgn}\, y$

No phase lag in $f(\cdot)$, $\arg G(i\omega) = -\pi$ for $\omega = \sqrt{3} = 1.7$

$$G(i\sqrt{3}) = -3/8 = -1/N(A) = -\pi A/4 \quad \Rightarrow \quad A = 12/8\pi \approx 0.48$$

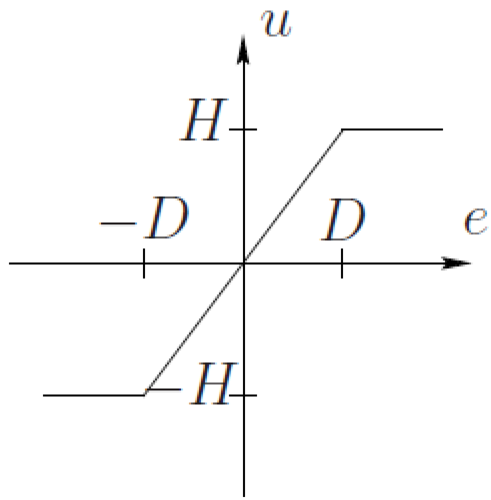
Simulation

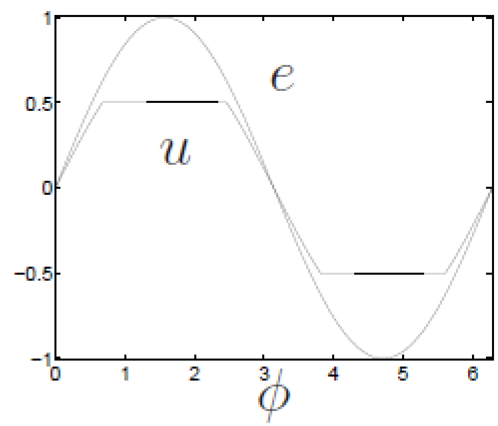
The prediction via the describing function agrees very well with the true oscillations:



Note that G filters out almost all higher-order harmonics.

Describing Function for a Saturation





Let $e(t) = A\sin\omega t = A\sin\phi$. First set $H = D$. Then for $\phi \in (0, \pi)$

$$u(\phi) = \begin{cases} A\sin\phi, & \phi \in (0, \phi_0) \cup (\pi - \phi_0, \pi) \\ D, & \phi \in (\phi_0, \pi - \phi_0) \end{cases}$$

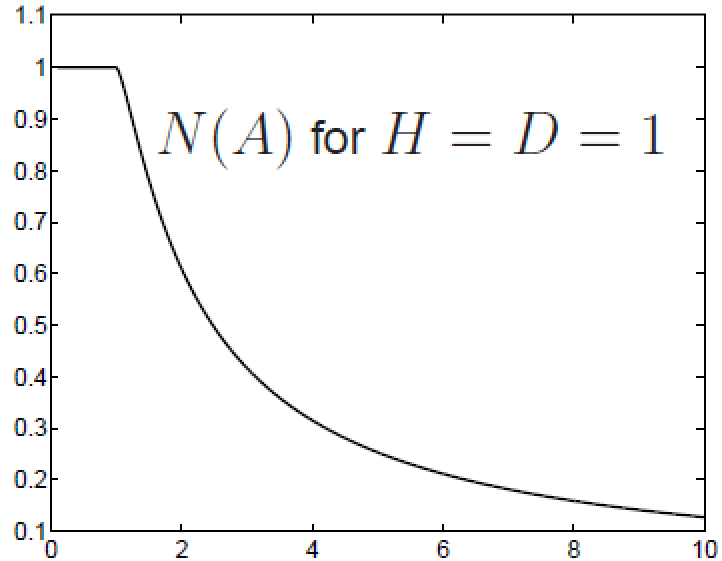
where $\phi_0 = \arcsin D/A$.

$$
\begin{aligned}
a_1 &= \frac{1}{\pi}\int_0^{2\pi} u(\phi)\cos\phi\, d\phi = 0 \\
b_1 &= \frac{1}{\pi}\int_0^{2\pi} u(\phi)\sin\phi\, d\phi = \frac{4}{\pi}\int_0^{\pi/2} u(\phi)\sin\phi\, d\phi \\
&= \frac{4A}{\pi}\int_0^{\phi_0} \sin^2\phi\, d\phi + \frac{4D}{\pi}\int_{\phi_0}^{\pi/2} \sin\phi\, d\phi \\
&= \frac{A}{\pi}\left(2\phi_0 + \sin 2\phi_0\right)
\end{aligned}
$$

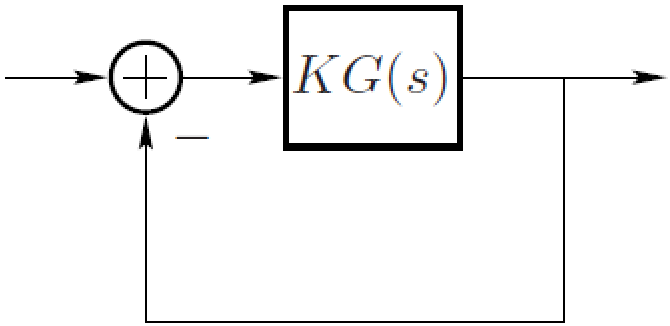
Hence, if $H = D$, then $N(A) = \frac{1}{\pi}\left(2\phi_0 + \sin 2\phi_0\right)$.

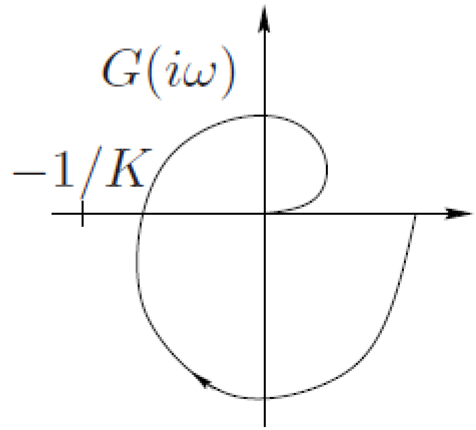
If $H \neq D$, then the rule $N_{\alpha f}(A) = \alpha N_f(A)$ gives

$$N(A) = \frac{H}{D\pi}\left(2\phi_0 + \sin 2\phi_0\right)$$



The Nyquist Theorem

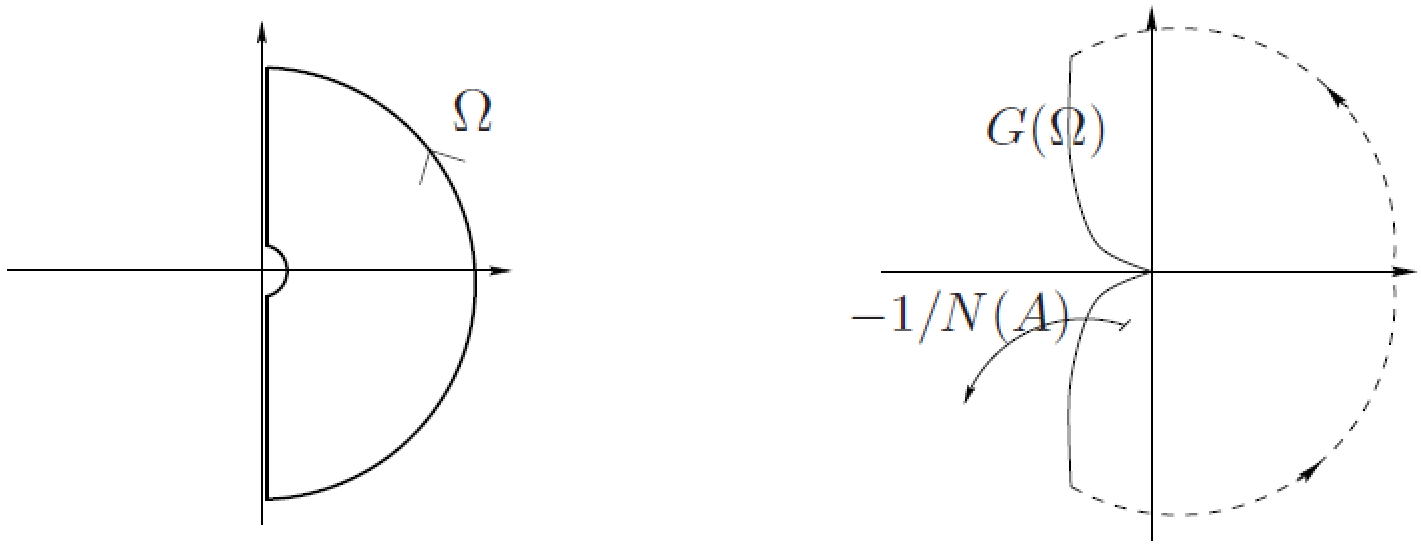




Assume that G is stable, and K is a positive gain.

- If $G(i\omega)$ goes through the point $-1/K$ the closed-loop system displays sustained oscillations
- If $G(i\omega)$ encircles the point $-1/K$, then the closed-loop system is unstable (growing amplitude oscillations).
- If $G(i\omega)$ does not encircle the point $-1/K$, then the closed-loop system is stable (damped oscillations)

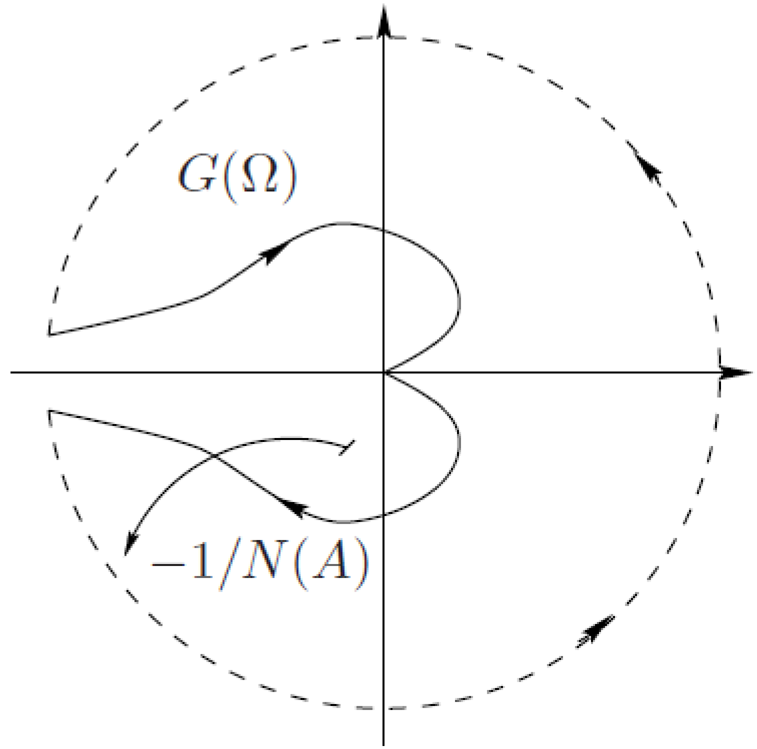
Stability of Periodic Solutions



Assume that $G(s)$ is stable.

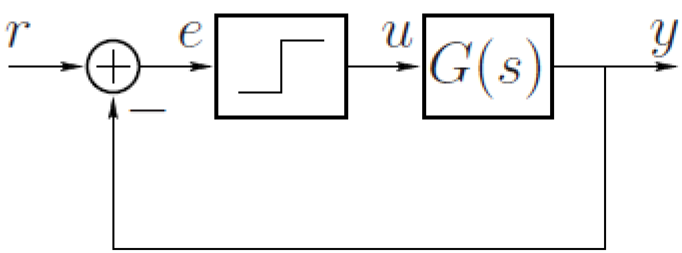
- If $G(\Omega)$ encircles the point $-1/N(A)$, then the oscillation amplitude is increasing.
- If $G(\Omega)$ does not encircle the point $-1/N(A)$, then the oscillation amplitude is decreasing.

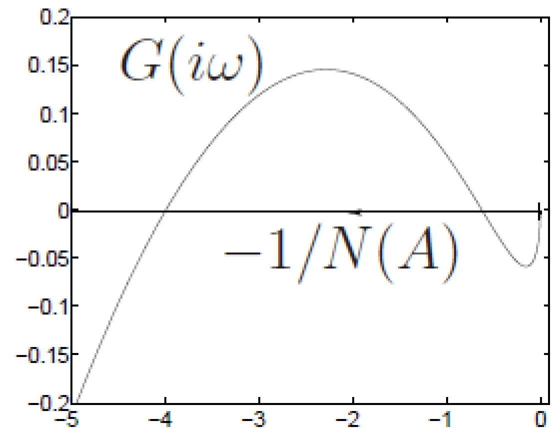
An Unstable Periodic Solution



An intersection with amplitude A_0 is unstable if $A < A_0$ leads to decreasing amplitude and $A > A_0$ leads to increasing.

Stable Periodic Solution in Relay System





$$G(s) = \frac{(s+10)^2}{(s+1)^3} \quad \text{with feedback} \quad u = -\text{sgn}\, y$$

gives one stable and one unstable limit cycle. The left most intersection corresponds to the stable one.

Fuzzy logic and fuzzy control

- Many plants are manually controlled by experienced operators
- Transfer process knowledge to control algorithm is difficult

Idea:

- Model operator's control actions (instead of the plant)
- Implement as rules (instead of as differential equations)

Example of a rule:

```
IF Speed is High AND Traffic is Heavy
    THEN Reduce Gas A Bit
```

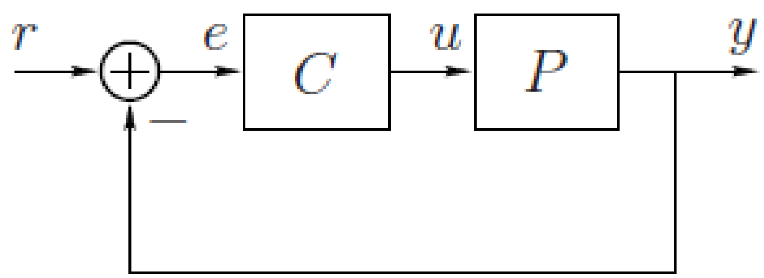
Model Controller Instead of Plant

Conventional control design

Model plant $P \longrightarrow$ Analyze feedback $\longrightarrow$ Synthesize controller $C \longrightarrow$ Implement control algorithm

Fuzzy control design

Model manual control $\longrightarrow$ Implement control rules



Fuzzy Set Theory

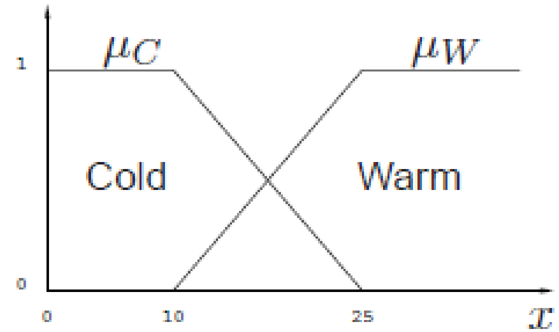
Specify how well an object satisfies a (vague) description

Conventional set theory: $x \in A$ or $x \notin A$

Fuzzy set theory: $x \in A$ to a certain degree $\mu_A(x)$

Membership function:

$\mu_A : \Omega \rightarrow [0, 1]$ expresses the degree x belongs to A

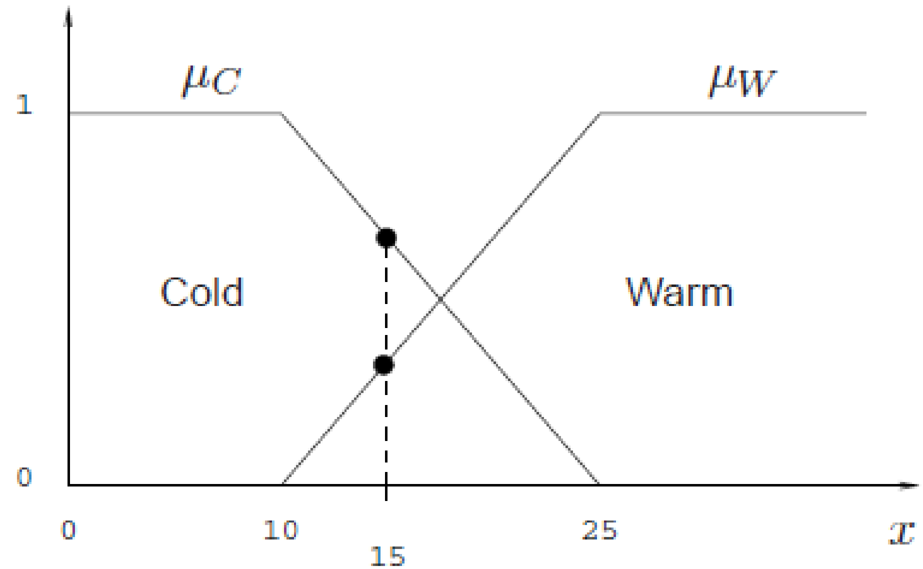


A **fuzzy set** is defined as (A, μ_A)

Example

Q1: Is the temperature $x = 15$ cold?
A1: It is quite cold since $\mu_C(15) = 2/3$.

Q2: Is $x = 15$ warm?
A2: It is not really warm since $\mu_W(15) = 1/3$.



Fuzzy Logic

How to calculate with fuzzy sets (A, μ_A)?

Conventional logic:

AND: $A \cap B$
OR: $A \cup B$
NOT: A'

Fuzzy logic:

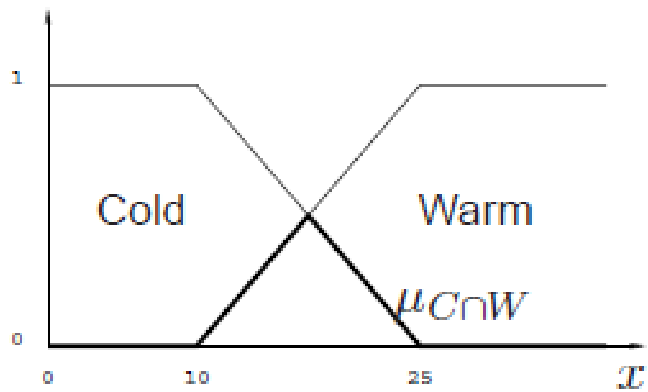
AND: $\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$
OR: $\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$
NOT: $\mu_{A'}(x) = 1 - \mu_A(x)$

Defines logic calculations as X AND Y OR Z

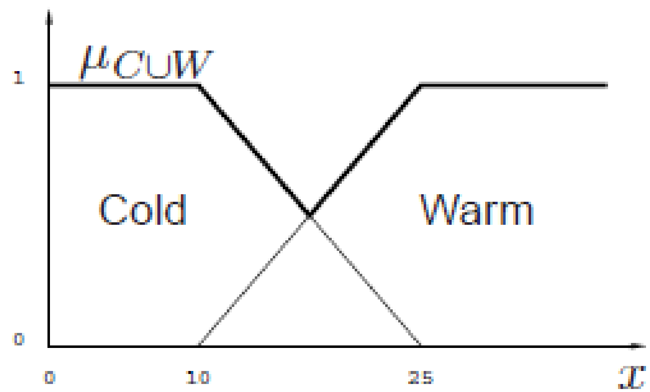
Mimic human linguistic (approximate) reasoning [Zadeh,1965]

Example

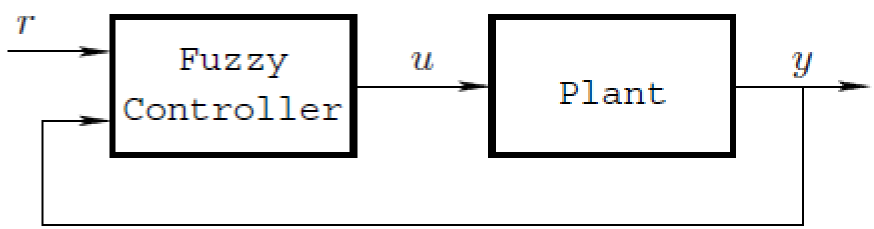
Q1: Is it cold AND warm?



Q2: Is it cold OR warm?



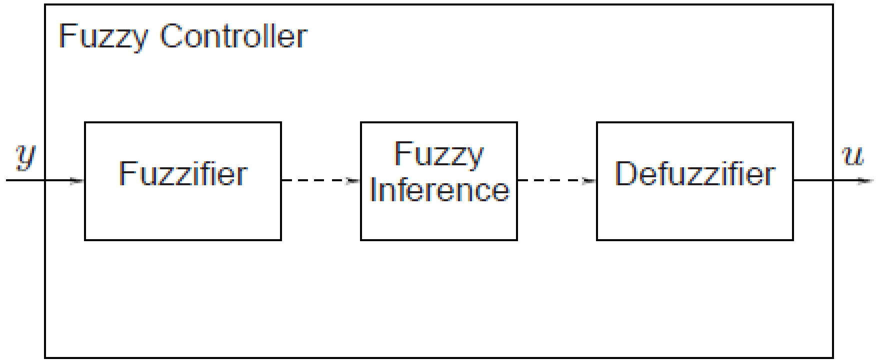
Fuzzy Control System



$r, y, u : [0, \infty) \mapsto \mathbb{R}$ are conventional signals

Fuzzy controller is a nonlinear mapping from y (and r) to u

Fuzzy Controller



Fuzzifier: Fuzzy set evaluation of y (and r)
Fuzzy Inference: Fuzzy set calculations
Defuzzifier: Map fuzzy set to u

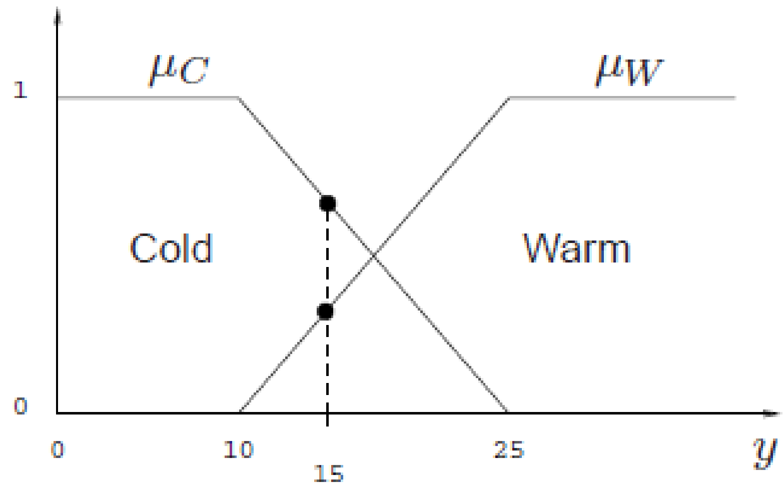
Fuzzifier and defuzzifier act as interfaces to the crisp signals

Fuzzifier

Fuzzy set evaluation of input y

Example

$y = 15$: $\mu_C(15) = 2/3$ and $\mu_W(15) = 1/3$



Fuzzy Inference

Fuzzy Inference:

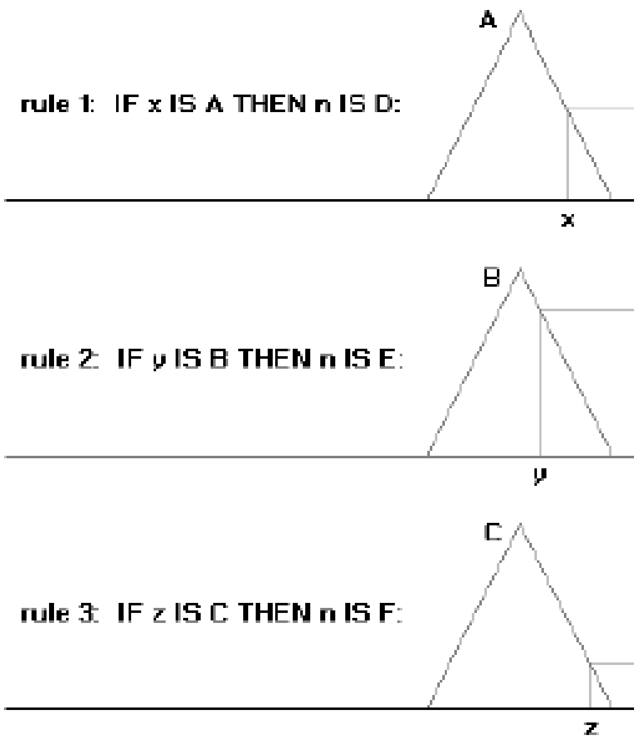
1. Calculate degree of fulfillment for each rule
2. Calculate fuzzy output of each rule
3. Aggregate rule outputs

Examples of fuzzy rules:

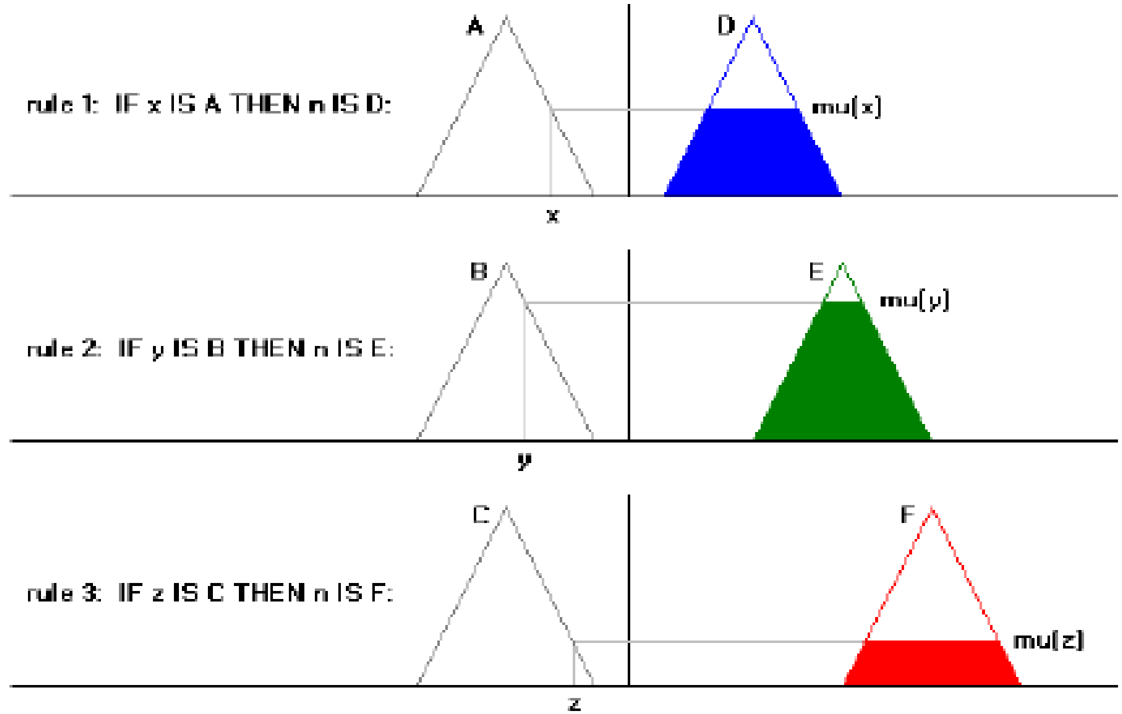
Rule 1: $\mathrm{IF}\ \underbrace{y \text{ is Cold}}_{1.}\ \mathrm{THEN}\ \underbrace{u \text{ is High}}_{2.}$

Rule 2: $\mathrm{IF}\ \underbrace{y \text{ is Warm}}_{1.}\ \mathrm{THEN}\ \underbrace{u \text{ is Low}}_{2.}$

1. Calculate degree of fulfillment for rules

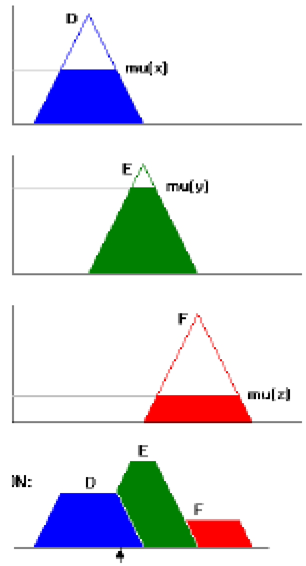


2. Calculate fuzzy output of each rule

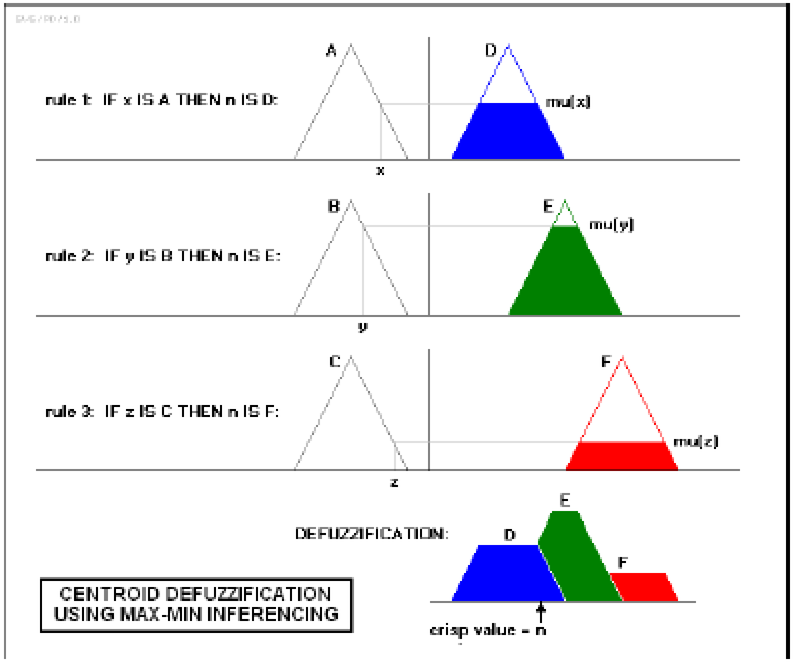


Note that "mu" is standard fuzzy-logic nomenclature for "truth value":

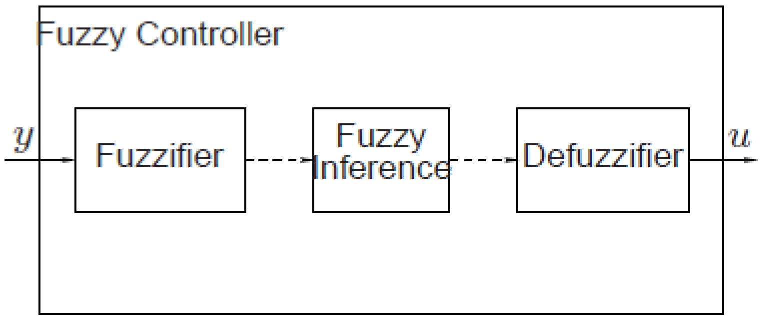
3. Aggregate rule outputs



Defuzzifier



Fuzzy Controller—Summary



Fuzzifier: Fuzzy set evaluation of y (and r)

Fuzzy Inference: Fuzzy set calculations

1. Calculate degree of fulfillment for each rule
2. Calculate fuzzy output of each rule
3. Aggregate rule outputs

Defuzzifier: Map fuzzy set to u

Pros and Cons of Fuzzy Control

Advantages

- User-friendly way to design nonlinear controllers
- Explicit representation of operator (process) knowledge
- Intuitive for non-experts in conventional control

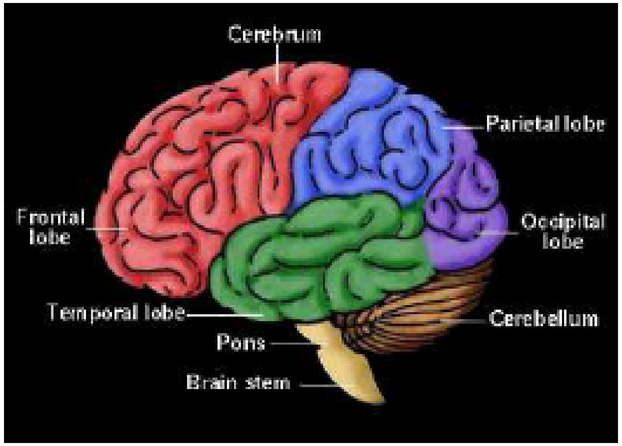
Disadvantages

- Limited analysis and synthesis
- Sometimes hard to combine with classical control
- Not obvious how to include dynamics in controller

Fuzzy control is a way to obtain a class of nonlinear controllers

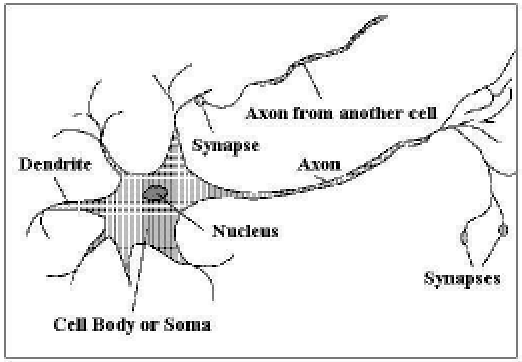
Neural Networks

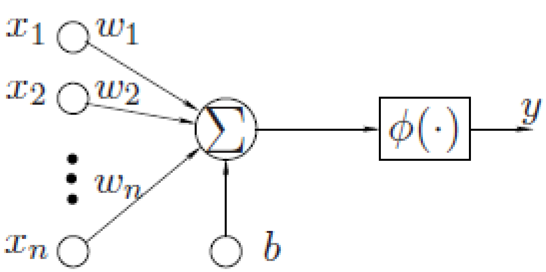
- How does the brain work?
- A network of computing components (neurons)



Neurons

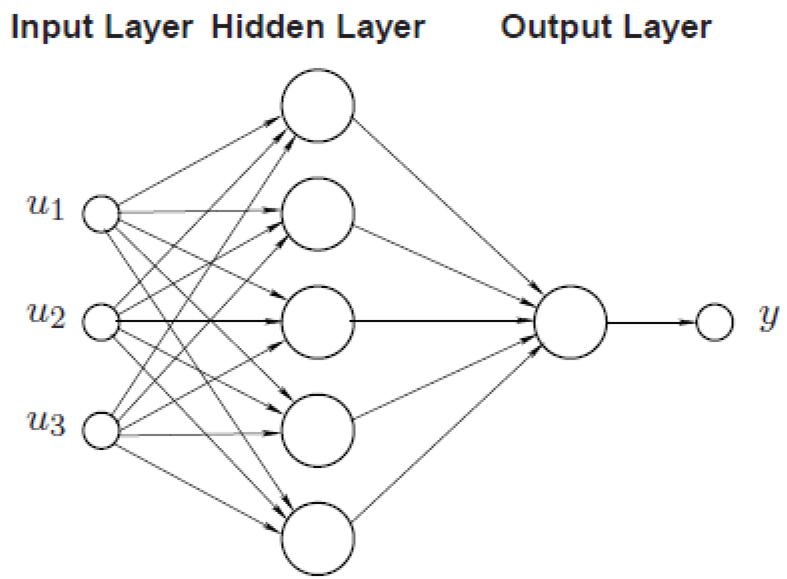
Brain neuron | **Artificial neuron**





A Simple Neural Network

Neural network consisting of six neurons:



Represents a nonlinear mapping from inputs to outputs

Neural Network Design

1. How many hidden layers?
2. How many neurons in each layer?
3. How to choose the weights?

The choice of weights are often done adaptively through **learning**

Success Stories

Fuzzy controls:

- Zadeh (1965)
- Complex problems but with possible linguistic controls
- Applications took off in mid 70's
  - Cement kilns, washing machines, vacuum cleaners

Artificial neural networks:

- McCulloch & Pitts (1943), Minsky (1951)
- Complex problems with unknown and highly nonlinear structure
- Applications took off in mid 80's
  - Pattern recognition (e.g., speech, vision), data classification