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Principle of Automatic Control (2)

自动控制原理2 (全英语教学课程)

Topic 2

Digital Control Systems

(Chapter 13 in text book)

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Learning Outcomes for Topic 1

After completing this topic, you will be able to:

- Model the digital computer in a feedback system;
- Find z- and inverse z-transforms of time and Laplace functions;
- Find sampled-data transfer functions;
- Reduce an interconnection of sampled-data transfer functions to a single sampled data transfer function;
- Determine whether a sampled-data system is stable and determine sampling rates for stability;
- Design digital systems to meet steady-state error specification;
- Design digital systems to meet transient response specifications using gain adjustment.

Outline

- Brief Introduction
- Modeling the Digital Computer
- The *z*-Transform
- Transfer Functions
- Block Diagram Reduction
- Stability
- Steady-State Errors
- Transient Response on the *z*-Plane

New terminologies in this topic

- Digital Control 数字控制
- Analog Signal 模拟信号
- Digital Signal 数字信号
- Analog-to-Digital (A/D) Converter 模数转换器
- Digital-to-Analog (D/A) Converter 数模转换器
- Instantaneous 同时的
- Distortion 失真
- Nyquist sampling rate 采样率
- Sampler 采样器
- Zero-Order Hold 零阶保持器
- Zero-Order Sample-and-Hold 零阶采样保持器
- Quantization Error 量化误差
- Unwieldy 不灵便的
- z-Transform z转换
- Power Series Method 幂级数法
- Synchronization 同步
- Phantom Sampler 幻影采样器
- Transcendental Equation 超越方程
- Bilinear Transformation 双线性变换

A timeline of control

Prehistory — Primitive — **Classical** — Modern

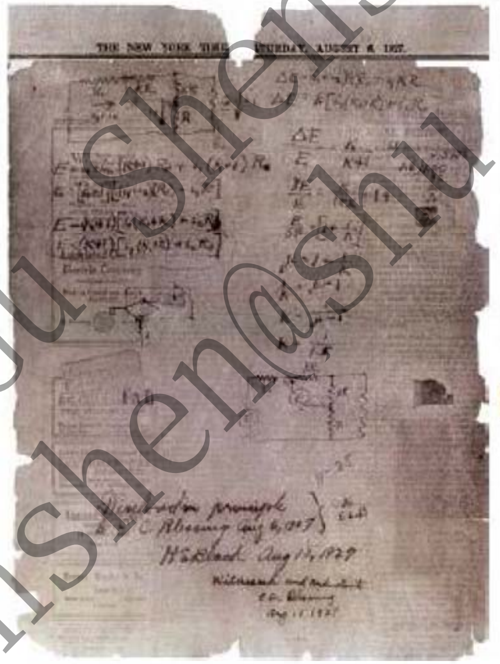
1868 — 1910 — WW2 — 1960 — 2012

Harry Nyquist
1889-1976

Nyquist plot

Harold Black
1898-1983

Negative feedback for electronic circuits



Hendrik Bode
1905-1982

Bode plot

Walter Evans
1920-1999

Root locus

We may call the period from 1868 to the early 1900's the *primitive period* of automatic control. It is standard to call the period from then until 1960 the *classical period*, and the period from 1960 through present times the *modern period*.

A timeline of control

Prehistory | Primitive | Classical | **Modern**

1868 | *1910* | *1960* | *2012*

space program

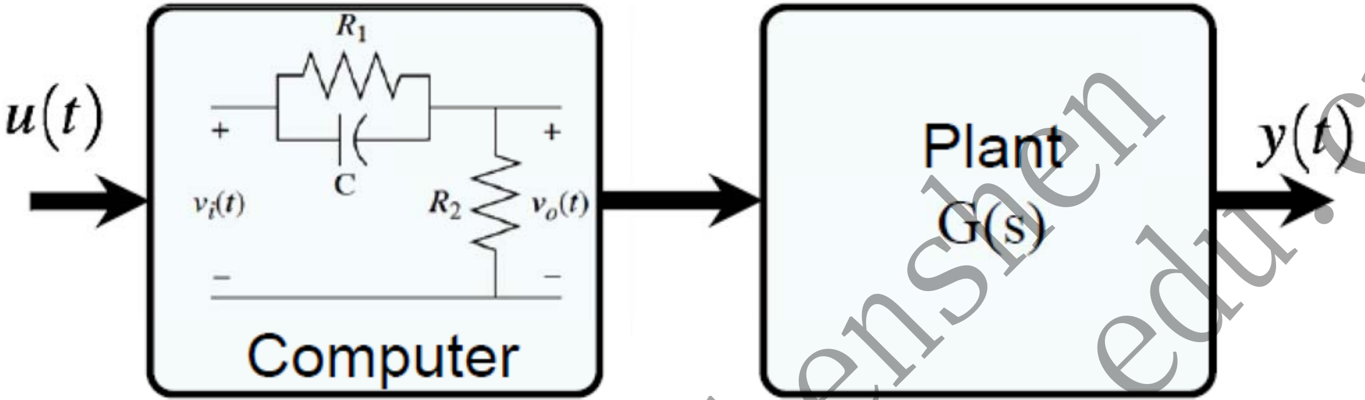
digital computer (late 40s)

State-space methods

Apollo lunar lander computer

1. Brief Introduction

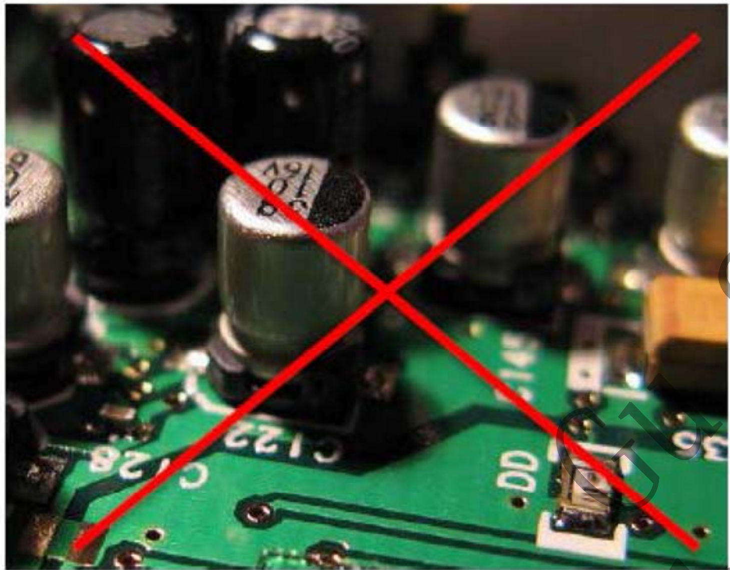
- The control design process
  - Obtain a model of the “plant” to be controlled, typically a transfer function expressed in “s”.
  - Choose the performance criteria that the controller must meet: settling time, overshoot etc.
  - Design a compensator using the method of your choice: root locus, Bode plot, etc.
  - Implement the compensator.
  - Evaluate the performance of the compensator against the design criteria. If there is not a good match then revisit steps 2-5.



$$G_c(s) = \frac{(s+4)}{s+20.09}$$

- It's simply a function
- Couldn't we just compute it?

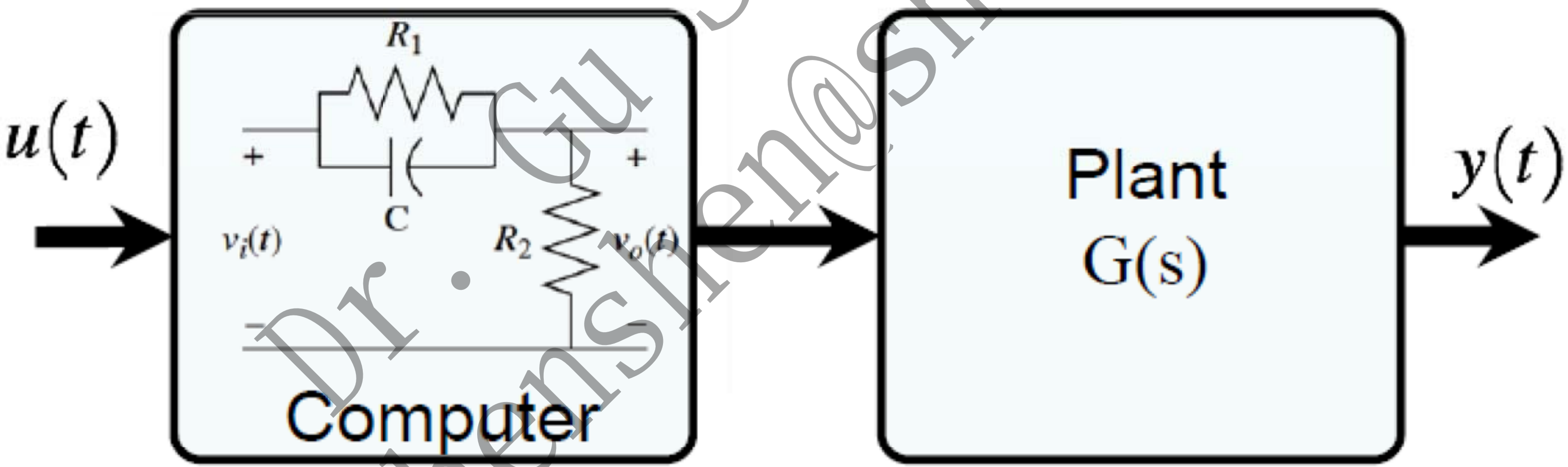
- Instead of building it with Rs and Cs



- we program it on a microcontroller

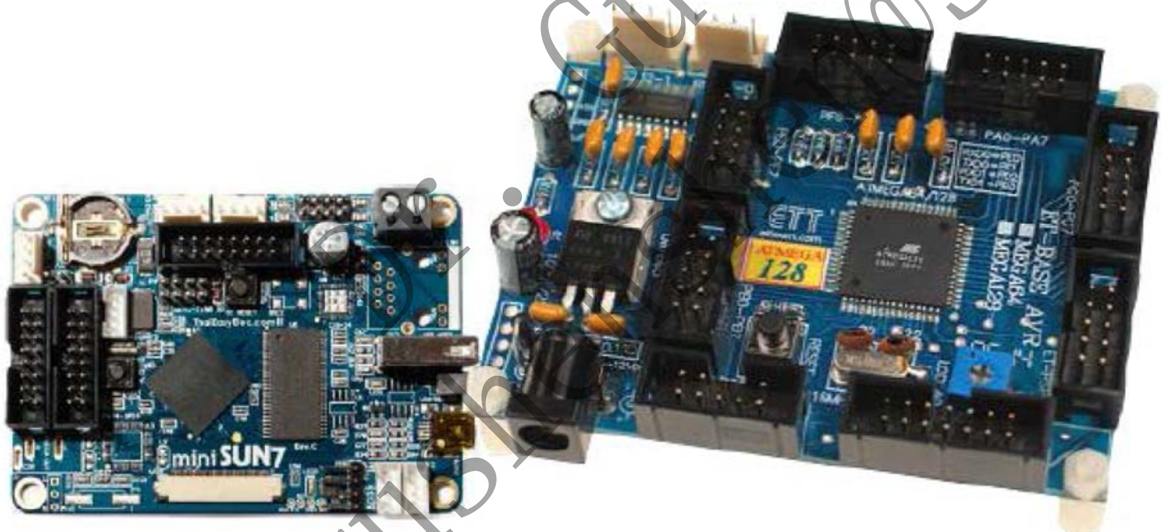
Discuss

- What are the advantages of doing it digitally?
- If u(t) and y(t) are analog signals how do we get them into a computer?



Why digital?

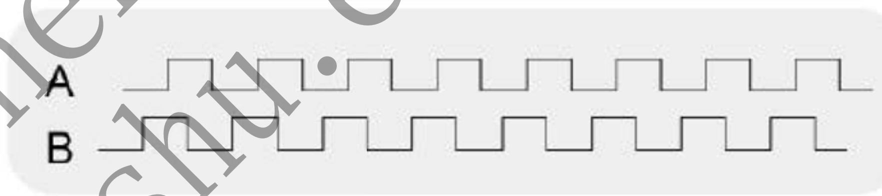
- It's a lot cheaper than you might think
- An 8-bit microprocessor costs $0.20 in quantity
- Software costs nothing to manufacture
- It can be changed after manufacture
  - "reflashing", changing the firmware
  - updating parameters in the firmware



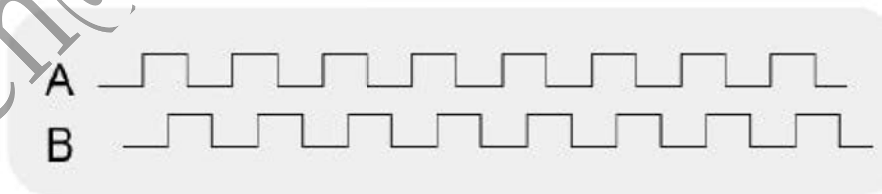
- you can add extra functionality quite easily
- for example a display
- or pushbuttons, mode selectors etc.

- or communications to a central monitoring or display station

- many sensors have digital, not analog, outputs

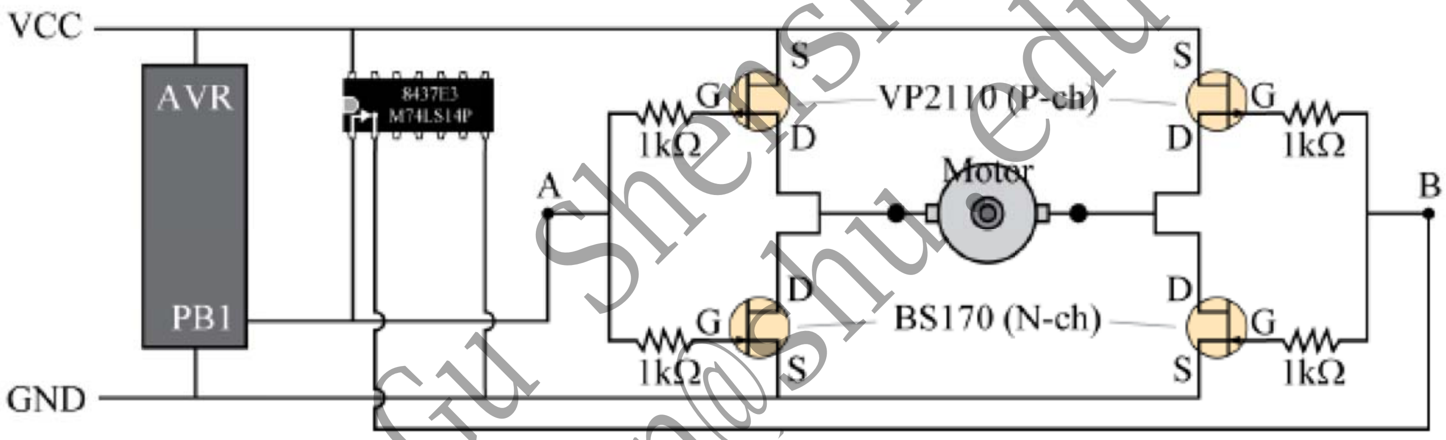


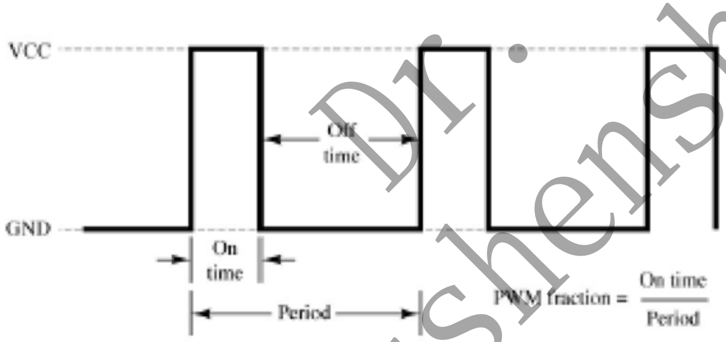
B leads A

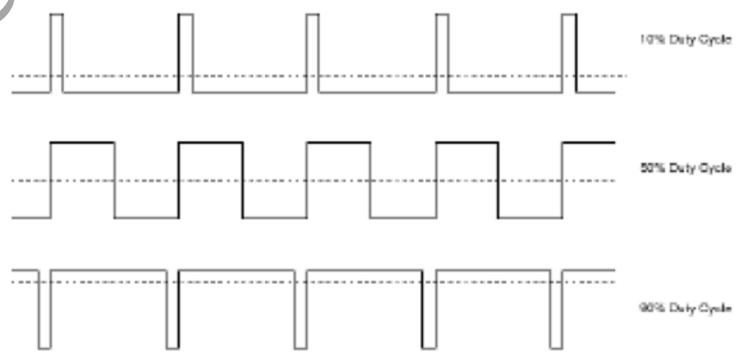


A leads B

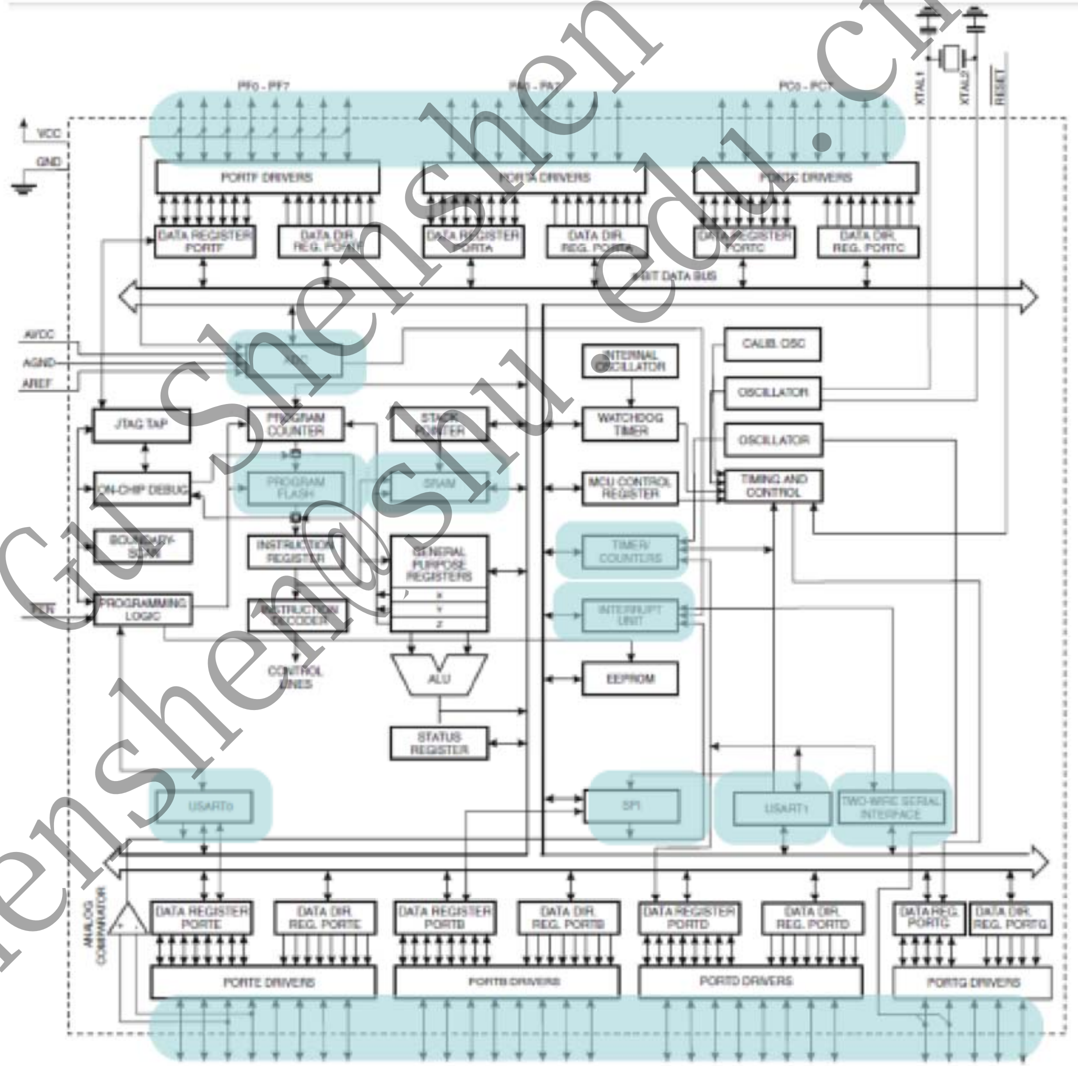
- Some systems are controlled by a digital input







Microcontrollers have many functions



Motor drives

- A really big application is controlling motors
  - Security cameras
  - ATMs
  - Printers
  - Hard drives
  - Vending machines
  - Cars
  - Manufacturing equipment: robots

Not all computers cost $0.2

$0.20

$300

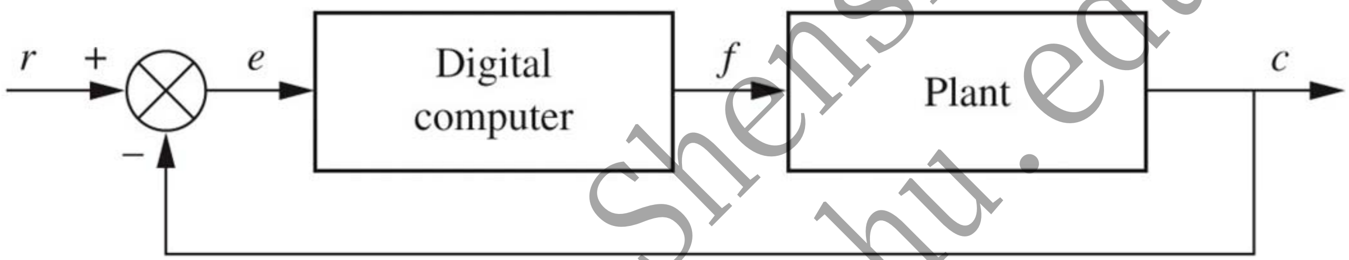
Discuss

- Why does the Intel processor cost more?

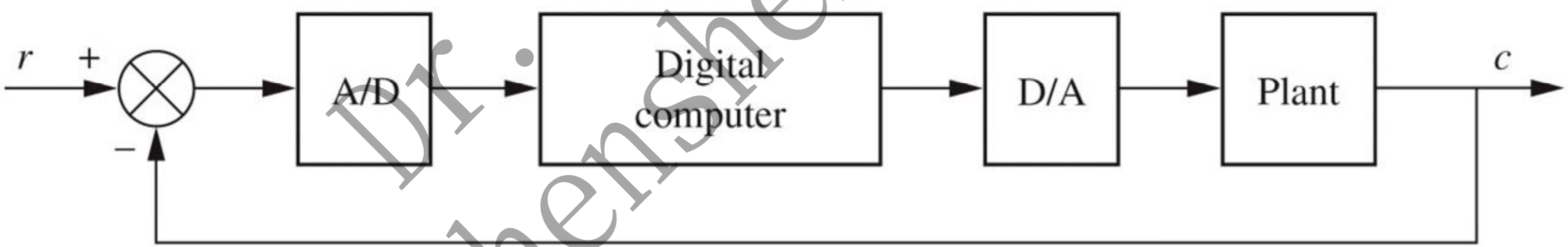
Partial list of answers

- It performs arithmetic much much faster
  - it's really 8 processors in one
- It can perform floating point arithmetic
- It can address a huge amount of memory

- The computer replaces the cascade compensator
- The signals can take on two forms: digital or analog.



(a)



(b)

Figure 13.2

A/D and D/A Converter

- A device that converts analog signals to digital signals is called an **analog-to-digital (A/D) converter**.
- Conversely, a device that converts digital signals to analog signals is called a **digital-to analog (D/A) converter.**

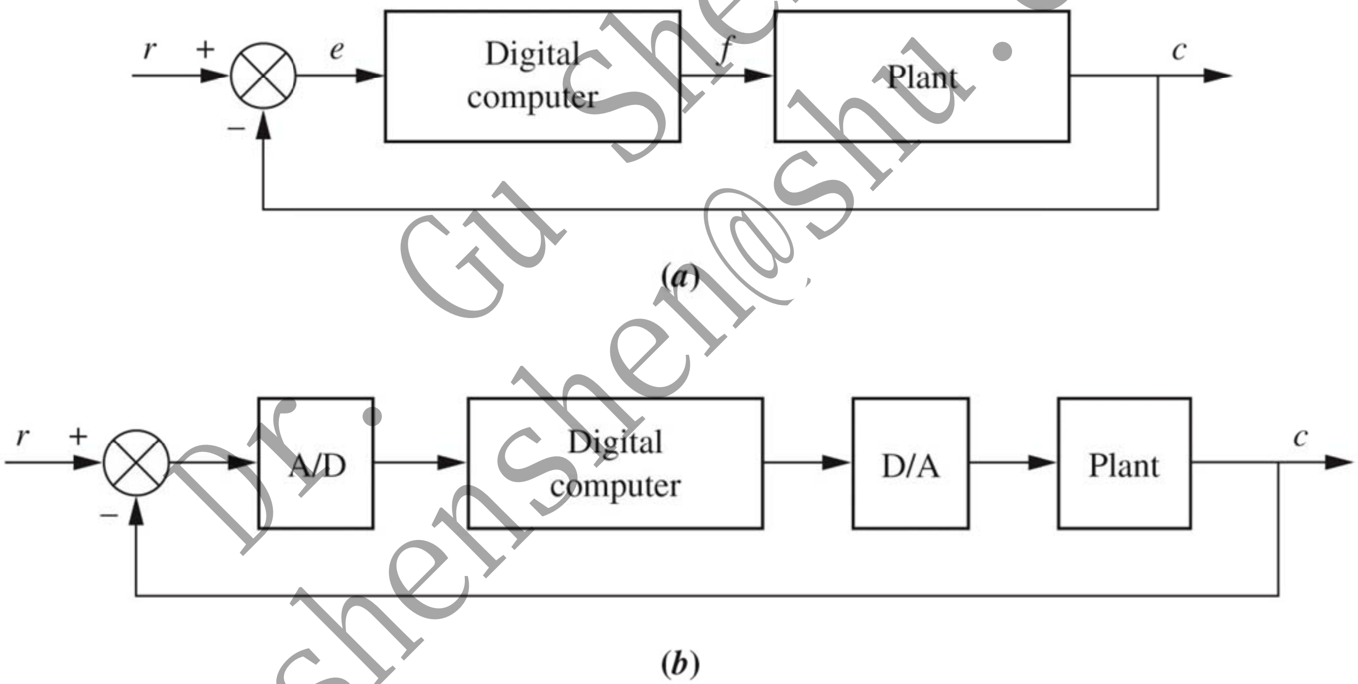


Figure 13.2

Digital-to-Analog Conversion

- Properly weighted voltages are summed together to yield the analog output.

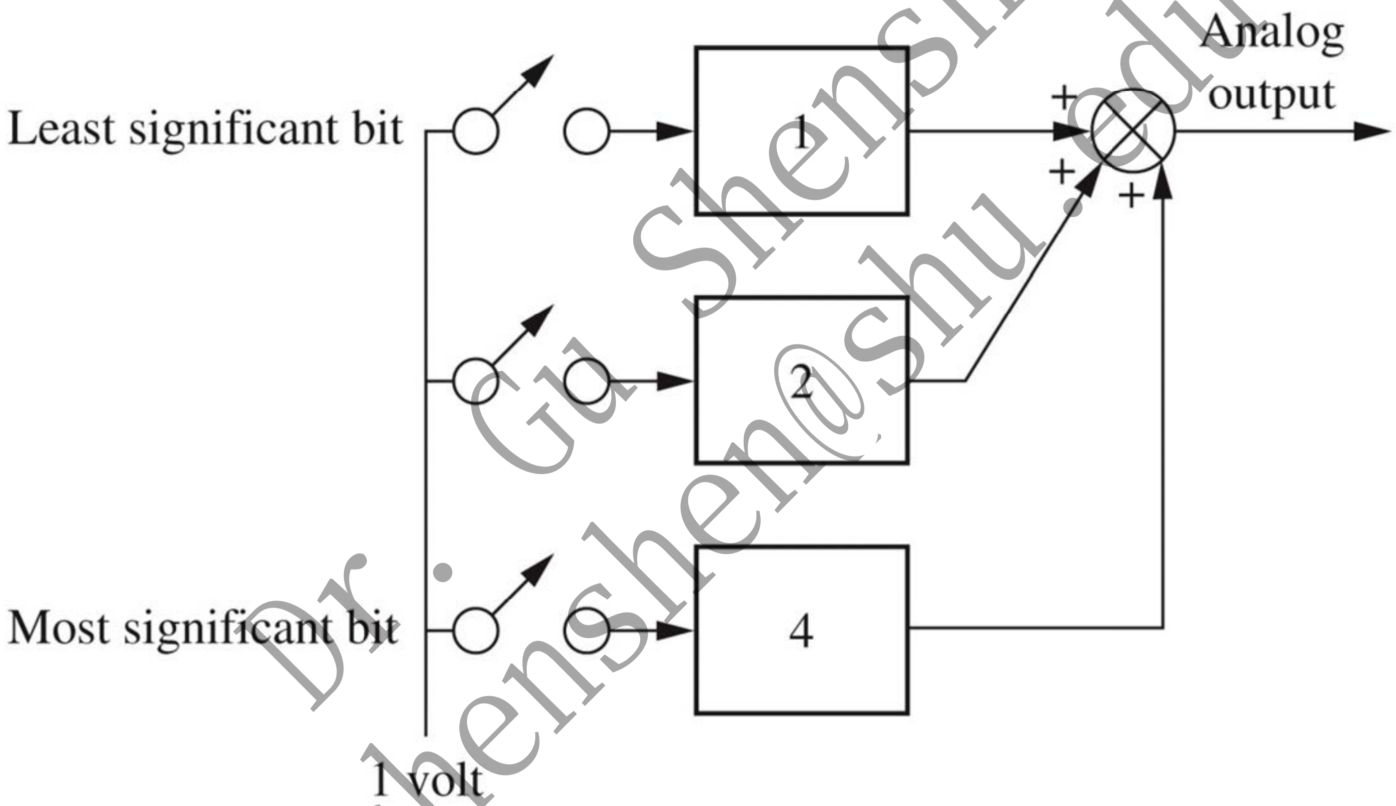
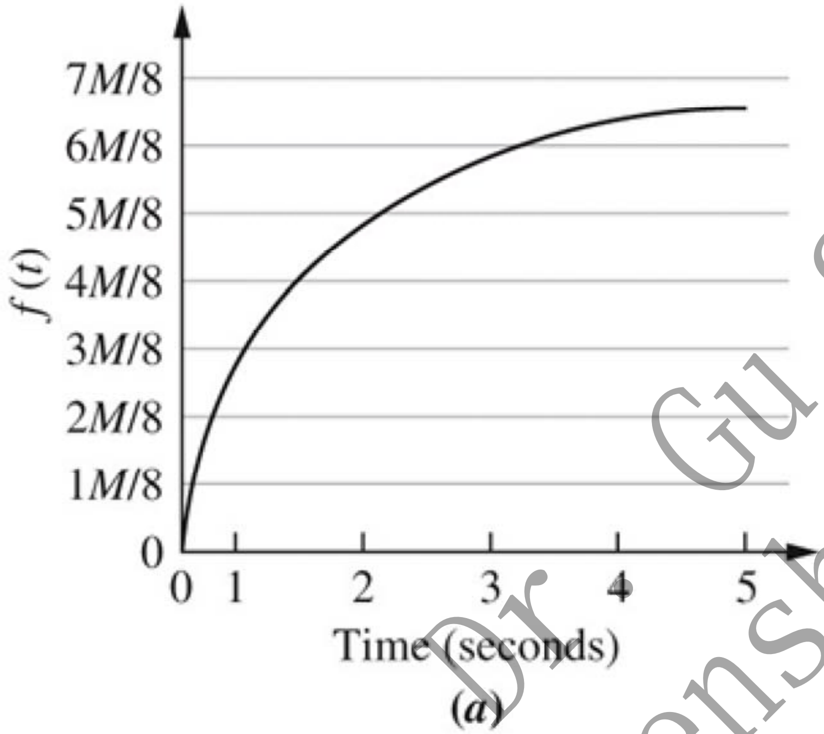


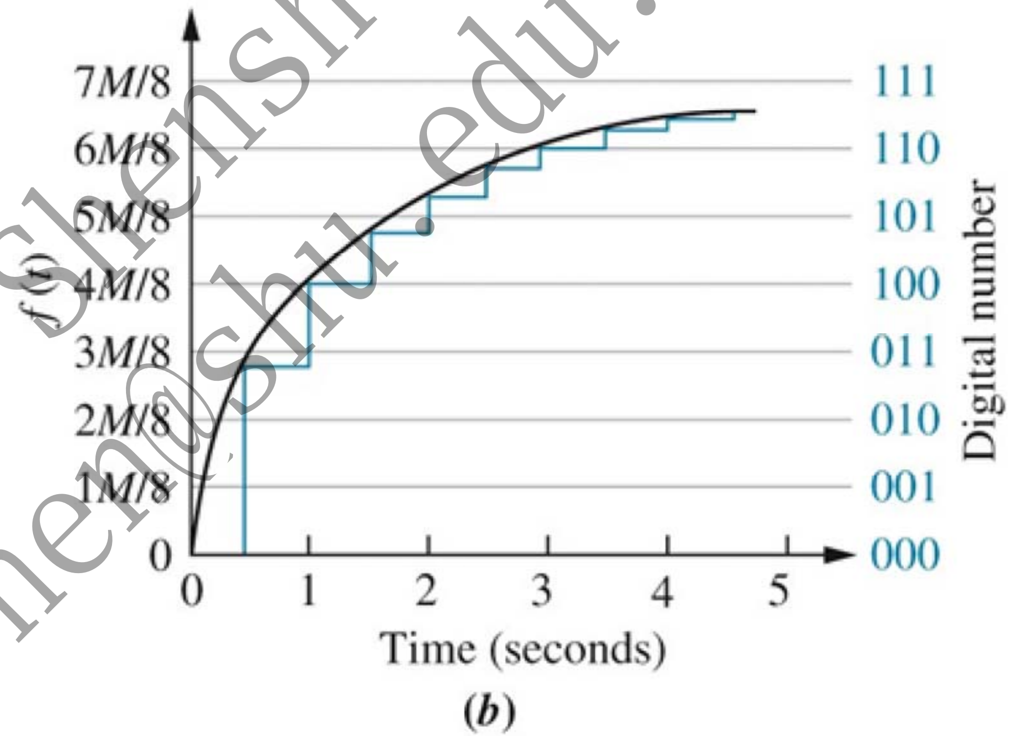
Figure 13.3

Analog-to-Digital Conversion

- In an analog-to-digital converter, the analog signal is first converted to a sampled signal and then converted to a sequence of binary numbers, the digital signal.
- The sampling rate must be at least twice the bandwidth of the signal, or else there will be distortion. This minimum sampling frequency is called the **Nyquist sampling rate**.
- The analog signal sampled at periodic intervals and held over the sampling interval by a device called a zero-order sample-and-hold (z.o.h.) that yields a staircase approximation to the analog signal.

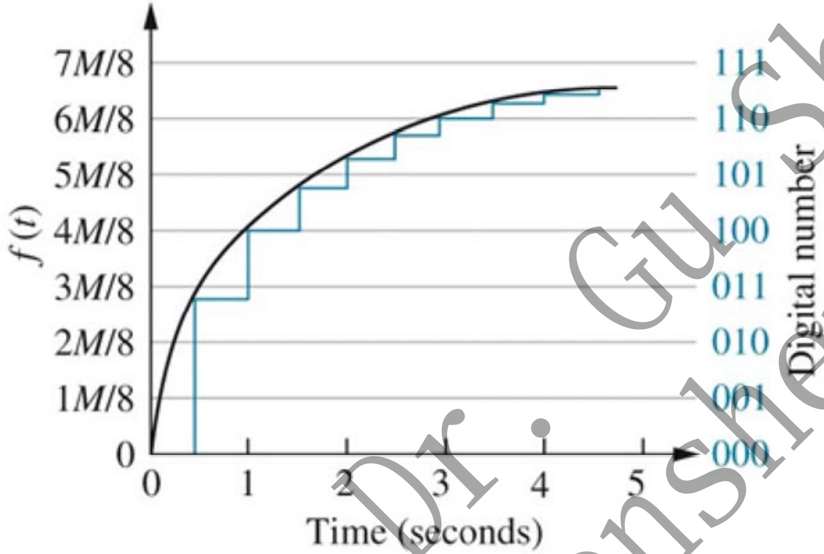


7M/8
6M/8
5M/8
4M/8
3M/8
2M/8
1M/8
0
f(t)
0 1 2 3 4 5
Time (seconds)
(a)

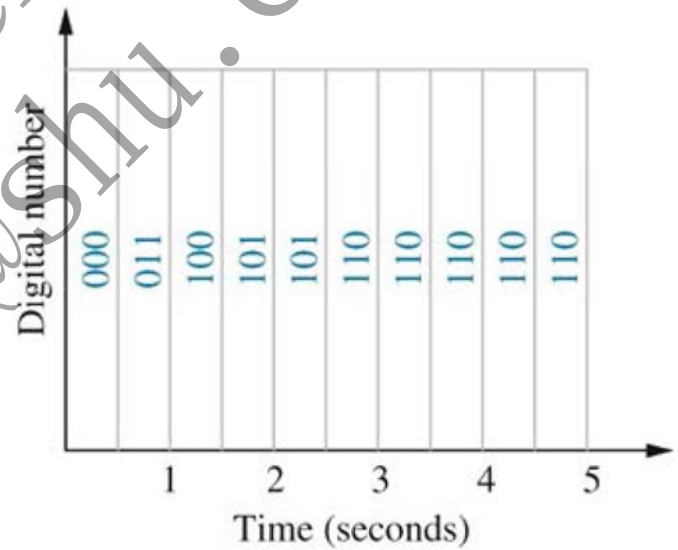


7M/8
6M/8
5M/8
4M/8
3M/8
2M/8
1M/8
0
f(t)
0 1 2 3 4 5
Time (seconds)
111
110
101
100
011
010
001
000
Digital number
(b)

- After sampling and holding, the analog-to-digital converter converts the sample to a digital number

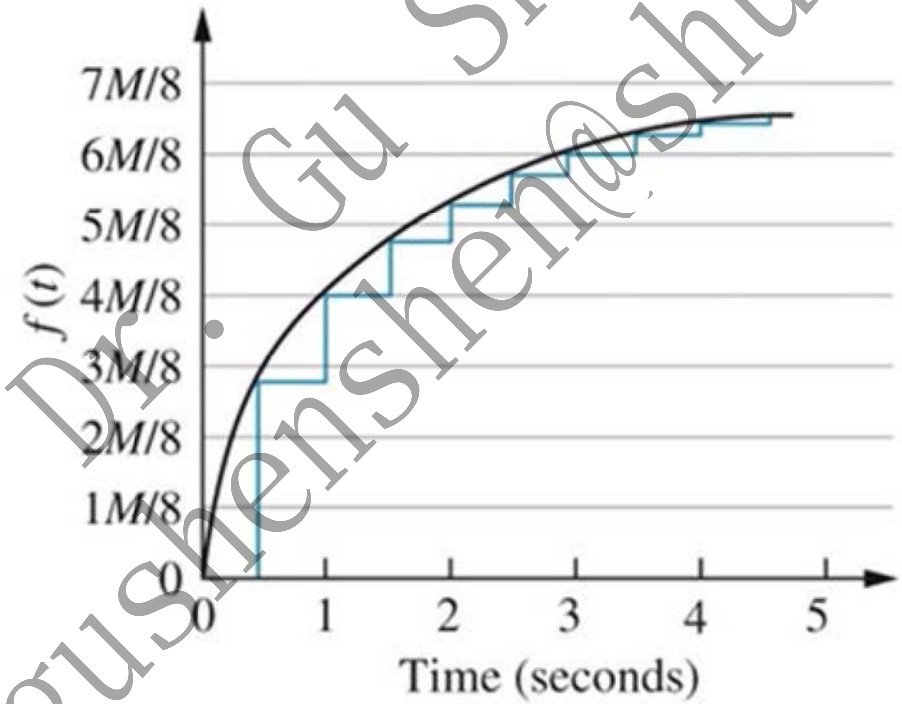


(b)



2. Modeling the Digital Computer

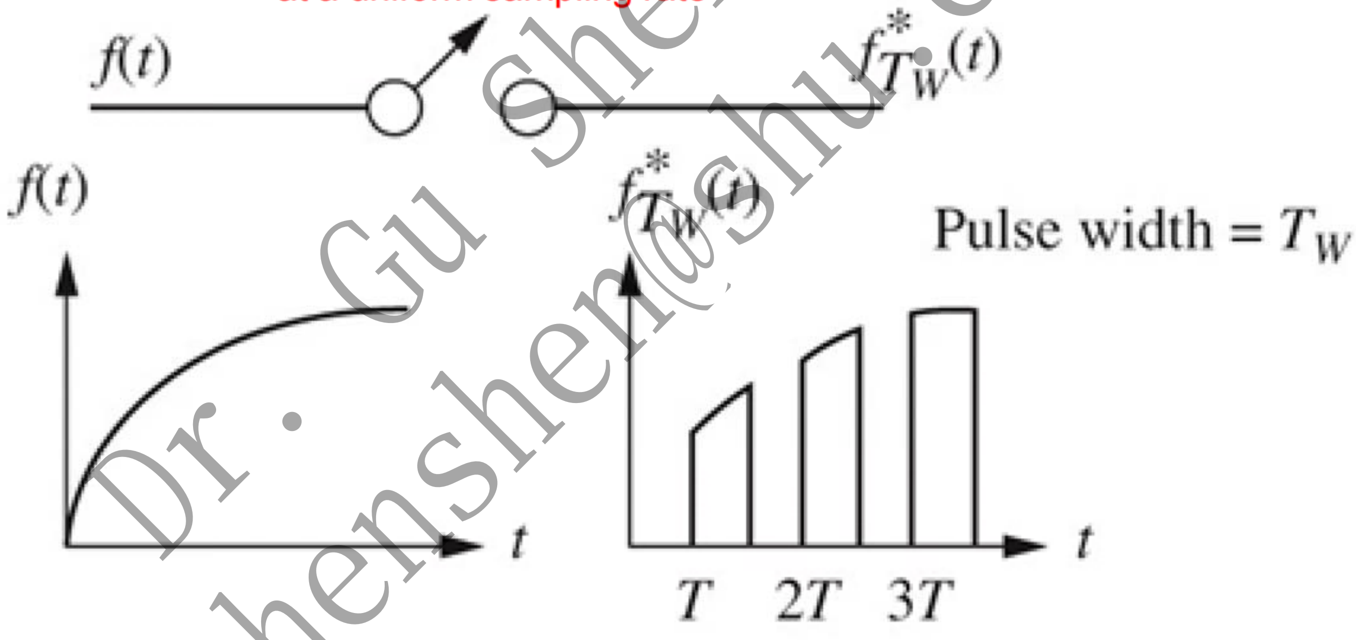
- The fact that signals are **sampled** at specified intervals and **held** causes the system performance to change with changes in **sampling rate**.
- The computer's effect upon the signal comes from this ***sampling and holding***.
- Thus, in order to model digital control systems, we must come up with a mathematical representation of this ***sample-and-hold process***.



Modeling the Sampler

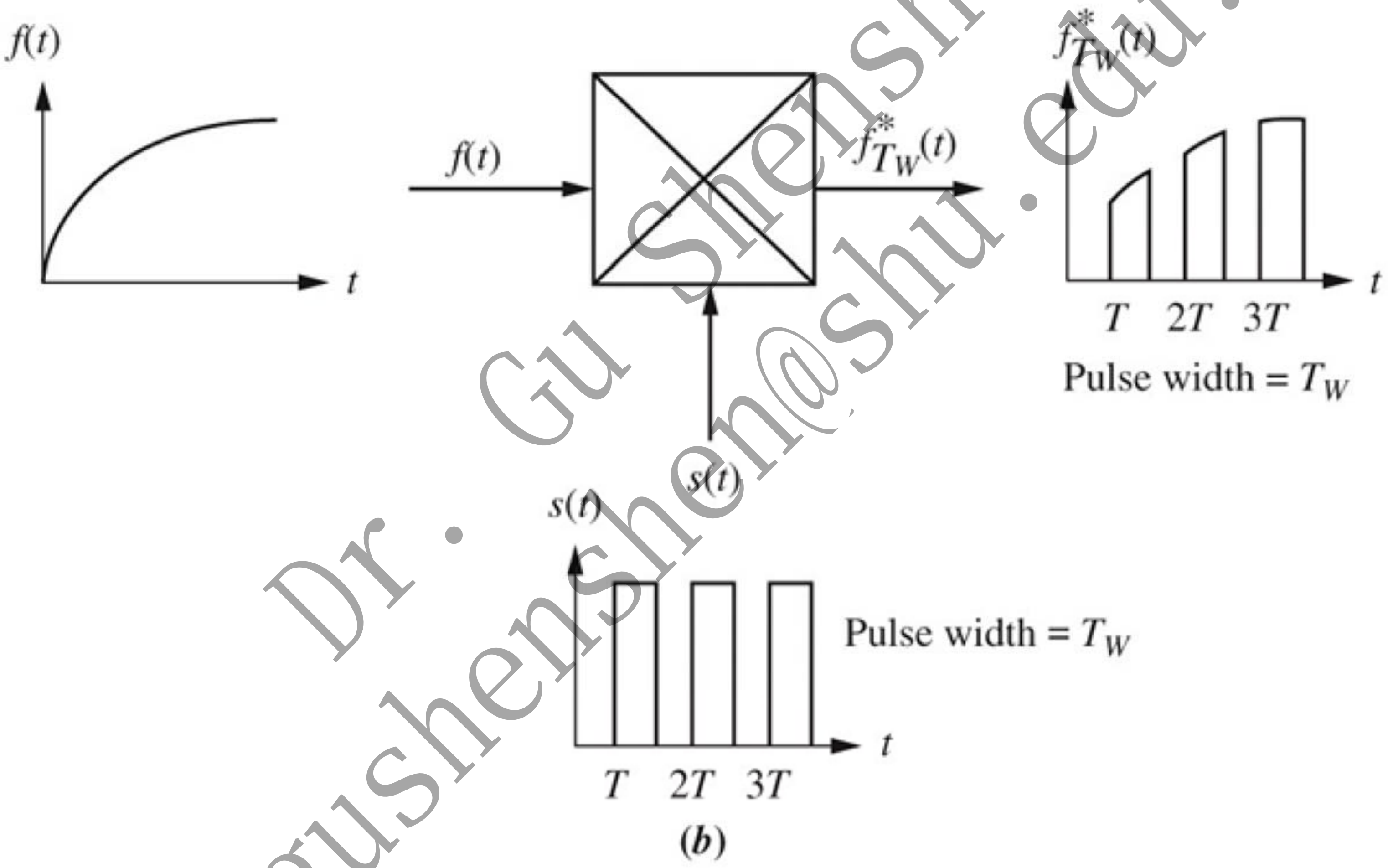
- Two views of uniform-rate sampling:
  - a. switch opening and closing;

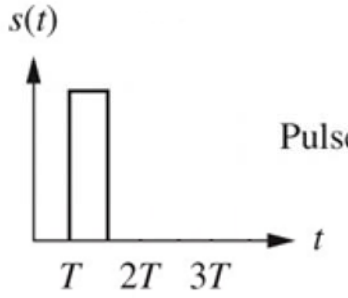
a switch turning on and off
at a uniform sampling rate



(a)

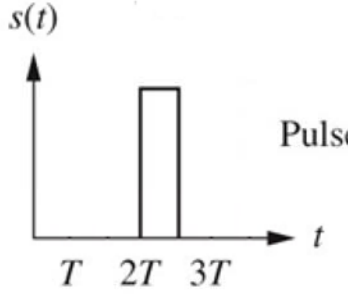
– b. product of time waveform and sampling waveform





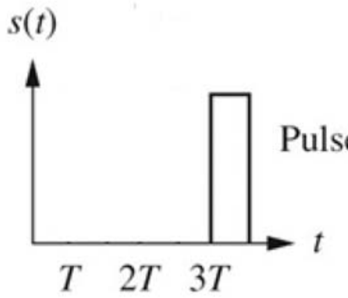
Pulse width = T_W

$$s(t)=u(t-T)-u(t-T-T_W)$$



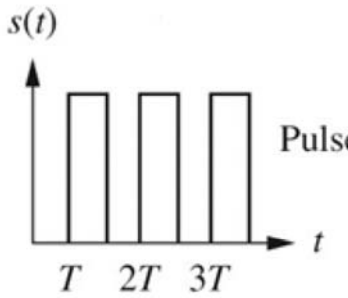
Pulse width = T_W

$$s(t)=u(t-2T)-u(t-2T-T_W)$$



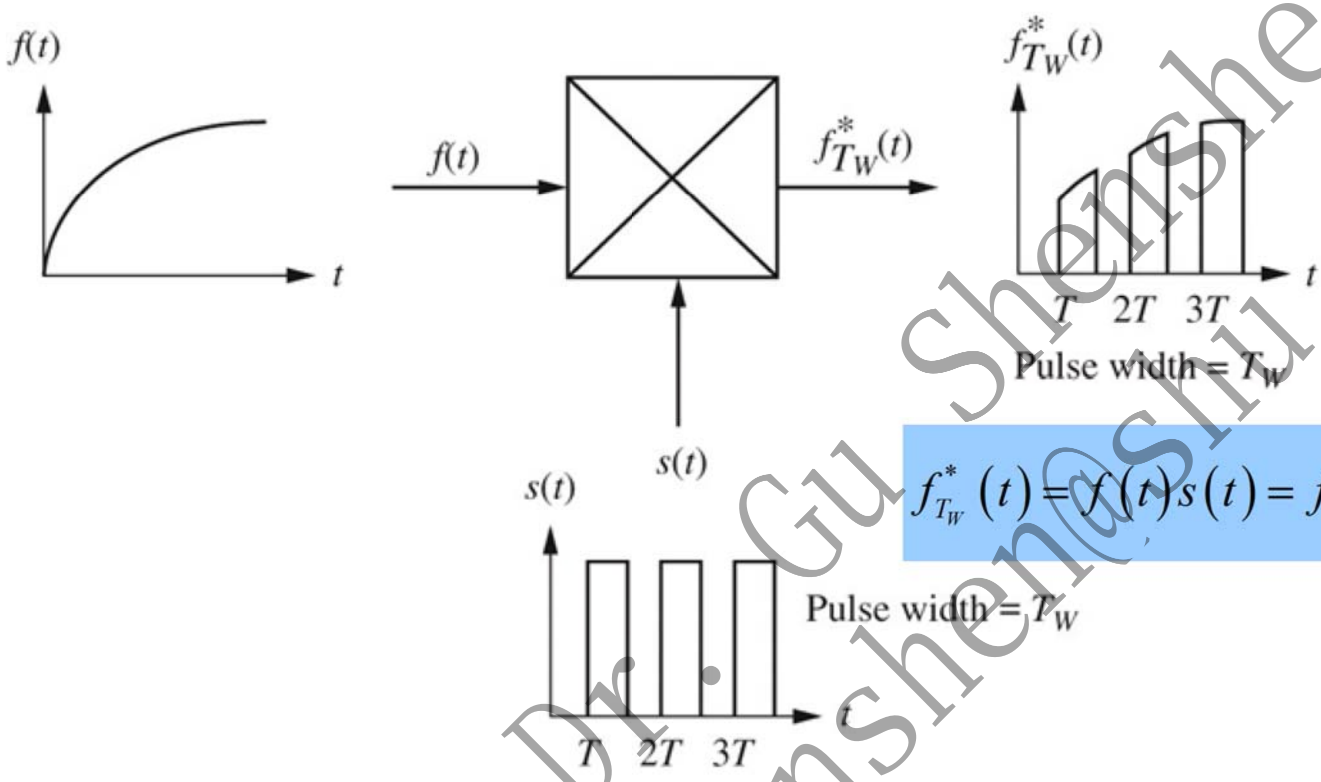
Pulse width = T_W

$$s(t)=u(t-3T)-u(t-3T-T_W)$$



Pulse width = T_W

$$s(t)=\sum_{k=-\infty}^{\infty}u(t-kT)-u(t-kT-T_W)$$



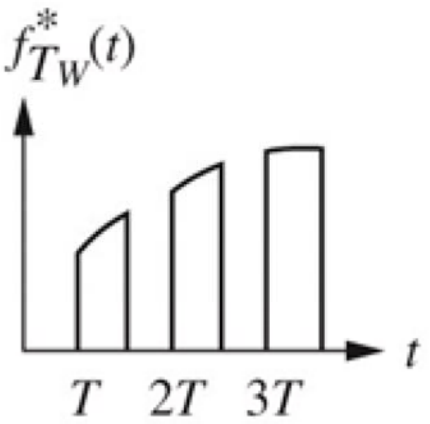
$$f_{T_W}^{*}(t)=f(t)s(t)=f(t)\sum_{k=-\infty}^{\infty}u(t-kT)-u(t-kT-T_W)$$

$$s(t)=\sum_{k=-\infty}^{\infty}u(t-kT)-u(t-kT-T_W)$$

Discuss: Can we apply Laplace transform?

$$f_{T_W}^{*}(t)=f(t)s(t)=f(t)\sum_{k=-\infty}^{\infty}u(t-kT)-u(t-kT-T_W)$$

It is the product of two time functions, taking the Laplace transform in order to find a transfer function is NOT simple.



Pulse width = T_W

$$f_{T_W}^{*}(t)=\sum_{k=-\infty}^{\infty}f(kT)\left[u(t-kT)-u(t-kT-Tw)\right]$$

A simplification can be made if we assume that the pulse width, *Tw*, is small in comparison to the period, *T*, such that *f(t)* can be considered constant during the sampling interval. Over the sampling interval, then, *f(t)=f(kT)*.

$$f_{T_W}^*(t)=\sum_{k=-\infty}^{\infty} f(kT)\left[u(t-kT)-u(t-kT-T_W)\right]$$

Taking the Laplace transform

$$F_{T_W}^*(s)=\sum_{k=-\infty}^{\infty} f(kT)\left[\frac{e^{-kTs}}{s}-\frac{e^{-kTs-T_Ws}}{s}\right]=\sum_{k=-\infty}^{\infty} f(kT)\left[\frac{1-e^{-T_Ws}}{s}\right]e^{-kTs}$$

Replacing $e^{T_W S}$ with its series expansion, we obtain

$$F_{T_W}^*(s)=\sum_{k=-\infty}^{\infty} f(kT)\left[\frac{1-\left\{1-T_Ws+\frac{(T_Ws)^2}{2!}\cdots\right\}}{s}\right]e^{-kTs}$$

For small T_W,

$$F_{Tw}^*(s)=\sum_{k=-\infty}^{\infty} f(kT)\left[\frac{T_Ws}{s}\right]e^{-kTs}=\sum_{k=-\infty}^{\infty} f(kT)T_We^{-kTs}$$

Finally, converting back to the time domain, we have

$$f_{T_W}^{*}(t) = T_W \sum_{k=-\infty}^{\infty} f(kT)\delta(t-kT)$$

$\delta(t-kT)$ Dirac delta function (unity pulse function)

Thus, the result of sampling with rectangular pulses can be thought of as a series of delta functions whose area is the product of the rectangular pulse width and the amplitude of the sampled waveform, or $T_W f(kT)$.

$$f_{T_W}^{*}(t)=T_W\sum_{k=-\infty}^{\infty} f(kT)\,\delta(t-kT)$$

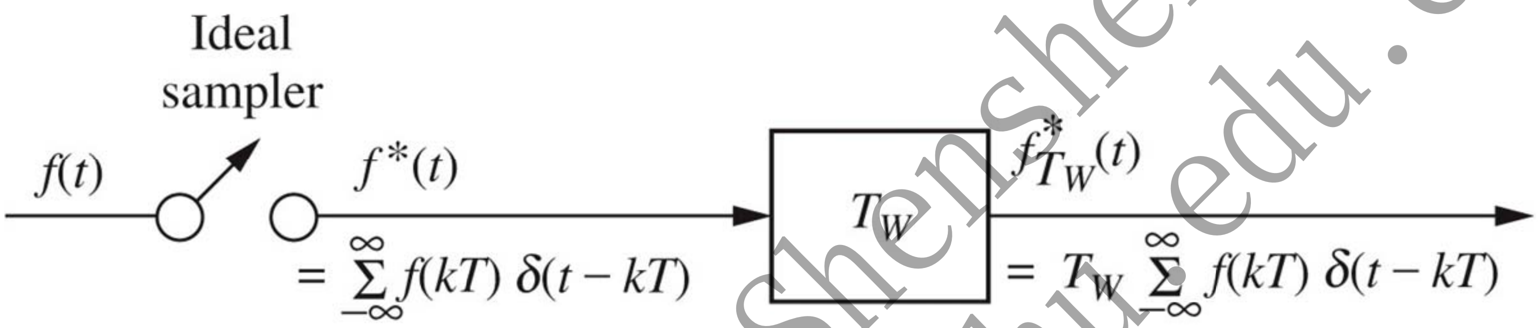


Figure 13.6

The sampler is divided into two parts:
(1) an **ideal sampler** described by the portion that is not dependent upon the sampling waveform characteristics,

$$f^{*}(t)=\sum_{k=-\infty}^{\infty} f(kT)\,\delta(t-kT)$$

and (2) the portion dependent upon the sampling waveform's characteristics, T_W.

Modeling the Zero-Order Hold

- The final step in modeling the digital computer is modeling the zero-order hold that follows the sampler.
- If we assume an ideal sampler (equivalent to setting T_W=1), then $f^*(t)$ is represented by a sequence of delta functions.
- The zero-order hold yields a staircase approximation to f(t).
- Hence, the output from the hold is a sequence of step functions whose amplitude is f(t) at the sampling instant, or *f(kT)*.

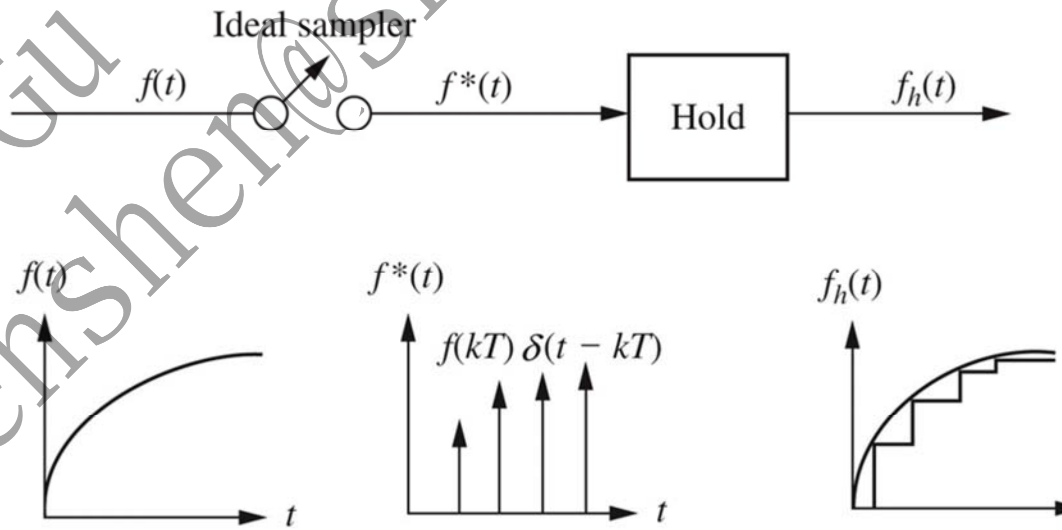
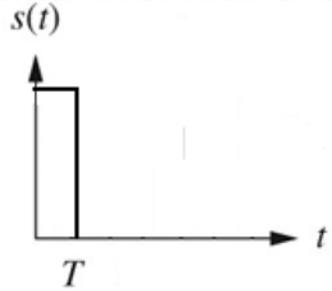
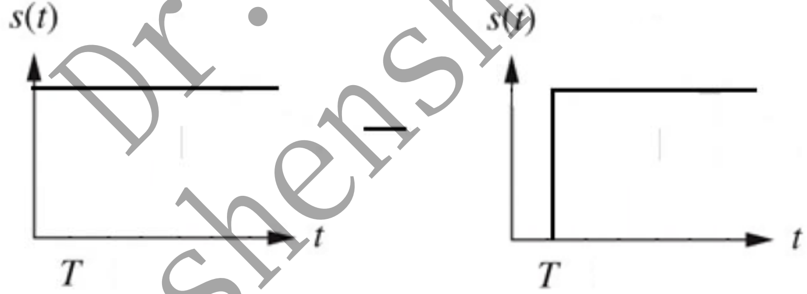


Figure 13.7

- The transfer function of any linear system is identical to the Laplace transform of the impulse response.
- Since a single impulse from the sampler yields a step over the sampling interval, the Laplace transform of this step, $G_h(s)$, which is the impulse response of the zero-order hold, is the transfer function of the zero-order hold.
- Using an impulse at zero time, the transform of the resulting step that starts at $t=0$ and ends at $t=T$ is



$$G_h(s)=\frac{1}{s}-\frac{1}{s}e^{-Ts}=\frac{1-e^{-Ts}}{s}$$



- Our goal is to develop a transform that contains the information of sampling from which sampled-data systems can be modeled with transfer functions, analyzed, and designed with the ease and insight we enjoyed with the Laplace transform.

$f^*(t)=\sum_{k=-\infty}^{\infty} f(kT)\delta(t-kT)$ is the ideal sampled waveform.

Taking the Laplace transform of this sampled time waveform, we obtain $F^*(s)=\sum_{k=0}^{\infty} f(kT)e^{-kTs}$

Now, letting $z=e^{Ts}$, the above equation can be written as $F(z)=\sum_{k=0}^{\infty} f(kT)z^{-k}$

This equation defines the **z-transform**. That is, an *F(z)* can be transformed to *f(kT)*, or an *f(kT)* can be transformed to *F(z)*.

$$f(kT)\rightleftarrows F(z)$$

Example 13.1

z-Transform of a Time Function

PROBLEM: Find the z-transform of a sampled unit ramp.

SOLUTION: For a unit ramp, $f(kT) = kT$. Hence the ideal sampled step can be written from Eq. (13.7) as

$$f^{*}t = \sum_{k=0}^{\infty} kt\,\delta(t - kT) \tag{13.12}$$

Taking the Laplace transform, we obtain

$$F^{*}(s) = \sum_{k=0}^{\infty} kTe^{-kTs} \tag{13.13}$$

Converting to the z-transform by letting $e^{-kTs} = z^{-k}$, we have

$$F(z) = \sum_{k=0}^{\infty} kTz^{-k} = T\sum_{k=0}^{\infty} kz^{-k} = T(z^{-1} + 2z^{-2} + 3z^{-3} + \cdots) \tag{13.14}$$

Equation (13.14) can be converted to a closed form by forming the series for $zF(z)$ and subtracting $F(z)$. Multiplying Eq. (13.14) by z, we get

$$zF(z) = T(1 + 2z^{-1} + 3z^{-2} + \cdots) \tag{13.15}$$

Subtracting Eq. (13.14) from Eq. (13.15), we obtain

$$zF(z) - F(z) = (z - 1)F(z) = T(1 + z^{-1} + z^{-2} + \cdots) \tag{13.16}$$

But

$$\frac{1}{1 - z^{-1}} = 1 + z^{-1} + z^{-2} + z^{-3} + \cdots \tag{13.17}$$

which can be verified by performing the indicated division. Substituting Eq. (13.17) into (13.16) and solving for $F(z)$ yields

$$F(z) = T\frac{z}{(z - 1)^2} \tag{13.18}$$

as the z-transform of $f(kT) = kT$.

table of z-transforms

TABLE 13.1 Partial table of z- and s-transforms

	$f(t)$	$F(s)$	$F(z)$	$f(kT)$
1.	$u(\mathrm{t})$	$\frac{1}{s}$	$\frac{z}{z-1}$	$u(kT)$
2.	t	$\frac{1}{s^2}$	$\frac{Tz}{(z-1)^2}$	kT
3.	t^n	$\frac{n!}{s^{n+1}}$	$\lim_{a\to 0}(-1)^n \frac{d^n}{da^n}\left[\frac{z}{z-e^{-aT}}\right]$	$(kT)^n$
4.	e^{-at}	$\frac{1}{s+a}$	$\frac{z}{z-e^{-aT}}$	e^{-akT}
5.	$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$	$(-1)^n \frac{d^n}{da^n}\left[\frac{z}{z-e^{-aT}}\right]$	$(kT)^n e^{-akT}$
6.	$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$	$\frac{z \sin \omega T}{z^2 - 2z\cos \omega T + 1}$	$\sin \omega kT$
7.	$\cos \omega t$	$\frac{s}{s^2+\omega^2}$	$\frac{z(z-\cos \omega T)}{z^2 - 2z\cos \omega T + 1}$	$\cos \omega kT$
8.	$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$	$\frac{ze^{-aT}\sin \omega T}{z^2 - 2ze^{-aT}\cos \omega T + e^{-2aT}}$	$e^{-akT}\sin \omega kT$
9.	$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$	$\frac{z^2 - ze^{-aT}\cos \omega T}{z^2 - 2ze^{-aT}\cos \omega T + e^{-2aT}}$	$e^{-akT}\cos \omega kT$

Table 13.1

z-transform theorems

TABLE 13.2 z-transform theorems

	Theorem	Name
1.	$z\{af(t)\} = aF(z)$	Linearity theorem
2.	$z\{f_1(t) + f_2(t)\} = F_1(z) + F_2(z)$	Linearity theorem
3.	$z\{e^{-aT}f(t)\} = F(e^{aT}z)$	Complex differentiation
4.	$z\{f(t - nT)\} = z^{-n}F(z)$	Real translation
5.	$z\{tf(t)\} = -Tz\frac{dF(z)}{dz}$	Complex differentiation
6.	$f(0) = \lim_{z\to\infty} F(z)$	Initial value theorem
7.	$f(\infty) = \lim_{z\to 1}(1 - z^{-1})F(z)$	Final value theorem

Table 13.2

The Inverse z-Transform

- Two methods for finding the inverse z-transform(the sampled time function from its z-transform):
  - (1) partial-fraction expansion and
  - (2) the power series method.

- Regardless of the method used, remember that since the z-transform came from the sampled waveform, the inverse z-transform will yield only the values of the time function at the sampling instants.

Inverse z-Transforms via Partial-Fraction Expansion

- Laplace transform consists of a partial fraction that yields a sum of terms leading to exponentials, that is, *A/(s+a).*
- Sampled exponential time functions are related to their z-transforms as follows:

$$e^{-akT} \rightleftarrows \frac{z}{z-e^{-aT}}$$

- We thus predict that a partial-fraction expansion should be of the following form:

$$F(z)=\frac{Az}{z-z_1}+\frac{Bz}{z-z_2}+\cdots$$

  – We first form *F(z)/z* to eliminate the z terms in the numerator;
  – Perform a partial-fraction expansion of *F(z)/z;*
  – Multiply the result by z to replace the z's in the numerator.

Example 13.2

Inverse z-Transform via Partial-Fraction Expansion

PROBLEM: Given the function in Eq. (13.21), find the sampled time function.

$$F(z) = \frac{0.5z}{(z-0.5)(z-0.7)} \qquad (13.21)$$

SOLUTION: Begin by dividing Eq. (13.21) by z and performing a partial-fraction expansion.

$$\frac{F(z)}{z} = \frac{0.5}{(z-0.5)(z-0.7)} = \frac{A}{z-0.5} + \frac{B}{z-0.7} = \frac{-2.5}{z-0.5} + \frac{2.5}{z-0.7} \qquad (13.22)$$

Next, multiply through by z.

$$F(z) = \frac{0.5z}{(z-0.5)(z-0.7)} = \frac{-2.5z}{z-0.5} + \frac{2.5z}{z-0.7} \qquad (13.23)$$

Using Table 13.1, we find the inverse z-transform of each partial fraction. Hence, the value of the time function at the sampling instants is

$$f(kT) = -2.5(0.5)^k + 2.5(0.7)^k \qquad (13.24)$$

Also, from Eqs. (13.7) and (13.24), the ideal sampled time function is

$$f^*(t) = \sum_{k=-\infty}^{\infty} f(kT)\delta(t-kT) = \sum_{k=-\infty}^{\infty} [-2.5(0.5)^k + 2.5(0.7)^k]\delta(t-kT) \qquad (13.25)$$

If we substitute $k = 0, 1, 2,$ and 3, we can find the first four samples of the ideal sampled time waveform. Hence,

$$f^*(t) = 0\delta(t) + 0.5\delta(t-T) + 0.6\delta(t-2T) + 0.545\delta(t-3T) \qquad (13.26)$$

Inverse z-Transform via the Power Series Method

- The values of the sampled time waveform can also be found directly from *F(z)*. Although this method does not yield closed-form expressions for *f (kT)*, it can be used for plotting.
- The method consists of performing the indicated division, which yields a power series for *F(z)*.

Example 13.3

Inverse z-Transform via Power Series

PROBLEM: Given the function in Eq. (13.21), find the sampled time function.

SOLUTION: Begin by converting the numerator and denominator of $F(z)$ to polynomials in z.

$$F(z) = \frac{0.5z}{(z-0.5)(z-0.7)} = \frac{0.5z}{z^2 - 1.2z + 0.35} \qquad (13.27)$$

Now perform the indicated division.

$$
\begin{array}{r l}
 & 0.5z^{-1} + 0.6z^{-2} + 0.545z^{-3} \\
\cline{2-2}
z^2 - 1.2z + 0.35 \,\big) & 0.5z \\
 & 0.5z - 0.6 + 0.175z^{-1} \\
\cline{2-2}
 & \qquad 0.6 - 0.175z^{-1} \\
 & \qquad 0.6 - 0.720z^{-1} + 0.21 \\
\cline{2-2}
 & \qquad\qquad 0.545z^{-1} - 0.21
\end{array}
\tag{13.28}
$$

Using the numerator and the definition of z, we obtain

$$F^*(s) = 0.5e^{-Ts} + 0.6e^{-2Ts} + 0.545e^{-3Ts} + \cdots \tag{13.29}$$

from which

$$f^*(t) = 0.5\delta(t - T) + 0.6\delta(t - 2T) + 0.545\delta(t - 3T) + \cdots \tag{13.30}$$

You should compare Eq. (13.30) with Eq. (13.26), the result obtained via partial expansion.

Skill-Assessment Exercise 13.1

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PROBLEM: Derive the z-transform for $f(t) = \sin \omega t \, u(t)$.

ANSWER: $$F(z) = \frac{z^{-1}\sin(\omega T)}{1 - 2z^{-1}\cos(\omega T) + z^{-2}}$$

The complete solution is located at www.wiley.com/college/nise.

13.1

$$f(t) = \sin(\omega kT);\ f^*(t) = \sum_{k=0}^{\infty} \sin(\omega kT)\delta(t-kT);$$

$$F^*(s) = \sum_{k=0}^{\infty} \sin(\omega kT)e^{-kTs} = \sum_{k=0}^{\infty} \frac{\left(e^{j\omega kT} - e^{-j\omega kT}\right)e^{-kTs}}{2j}$$

$$= \frac{1}{2j}\sum_{k=0}^{\infty} \left(e^{T(s-j\omega)}\right)^{-k} - \left(e^{T(s+j\omega)}\right)^{-k}$$

But, $\sum_{k=0}^{\infty} x^{-k} = \frac{1}{1-x^{-1}}$

Thus,

$$F^*(s) = \frac{1}{2j}\left[\frac{1}{1-e^{-T(s-j\omega)}} - \frac{1}{1-e^{-T(s+j\omega)}}\right] = \frac{1}{2j}\left[\frac{e^{-Ts}e^{j\omega T} - e^{-Ts}e^{j\omega T}}{1-\left(e^{-Ts}e^{j\omega T} - e^{-Ts}e^{j\omega T}\right) + e^{-2Ts}}\right]$$

$$= e^{-Ts}\left[\frac{\sin(\omega T)}{1 - e^{-Ts}2\cos(\omega T) + e^{-2Ts}}\right] = \frac{z^{-1}\sin(\omega T)}{1 - 2z^{-1}\cos(\omega T) + z^{-2}}$$

Skill-Assessment Exercise 13.2

PROBLEM: Find $f(kT)$ if $F(z) = \dfrac{z(z+1)(z+2)}{(z-0.5)(z-0.7)(z-0.9)}$.

ANSWER: $f(kT) = 46.875(0.5)^k - 114.75(0.7)^k + 68.875(0.9)^k$

The complete solution is located at www.wiley.com/college/nise.

13.2

$$F(z)=\frac{z(z+1)(z+2)}{(z-0.5)(z-0.7)(z-0.9)}$$

$$\frac{F(z)}{z}=\frac{(z+1)(z+2)}{(z-0.5)(z-0.7)(z-0.9)}$$

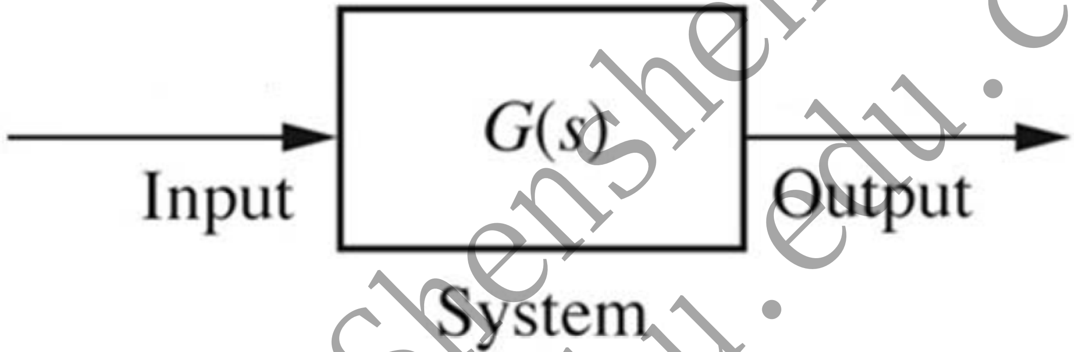
$$=46.875\frac{1}{z-0.5}-114.75\frac{1}{z-0.7}+68.875\frac{1}{z-0.9}$$

$$F(z)=46.875\frac{z}{z-0.5}-114.75\frac{z}{z-0.7}+68.875\frac{z}{z-0.9},$$

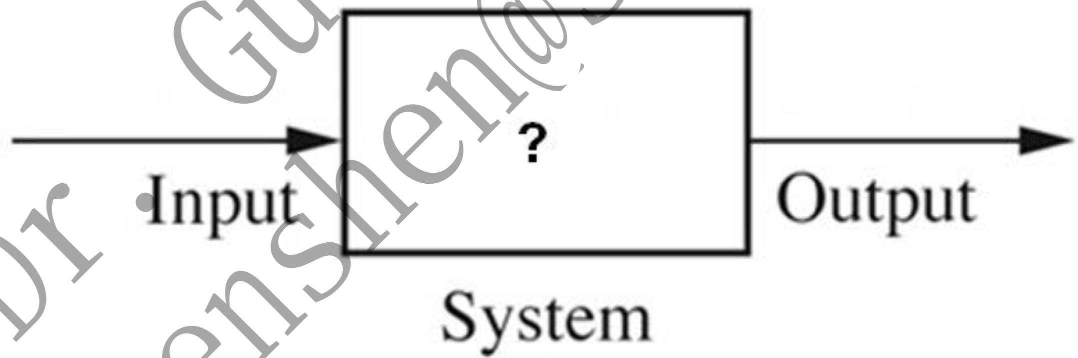
$$f(kT)=46.875(0.5)^k-114.75(0.7)^k+68.875(0.9)^k$$

4. Transfer Functions

Analog system



Digital system

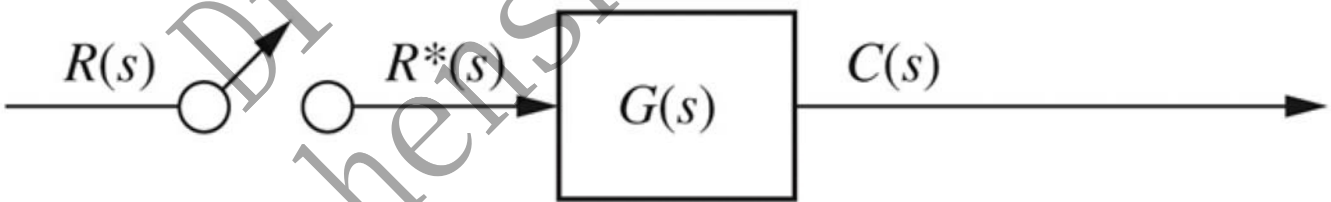


- Now that we have established the z-transform, let us apply it to physical systems by finding transfer functions of sampled-data systems.

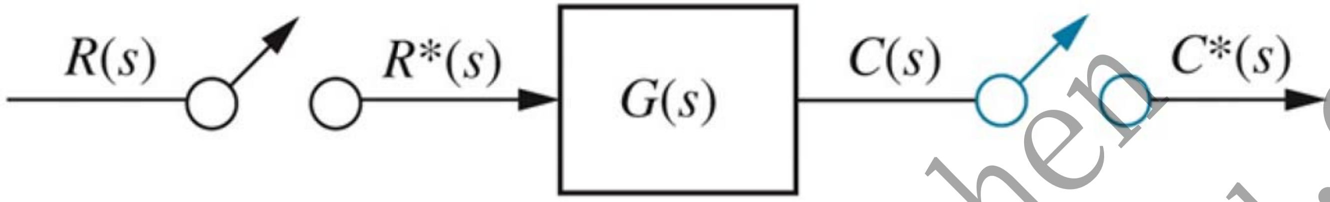
Consider the continuous system in Figure (a)

(a)

If the input is sampled as shown in Figure (b), the output is still a continuous signal.



(b)



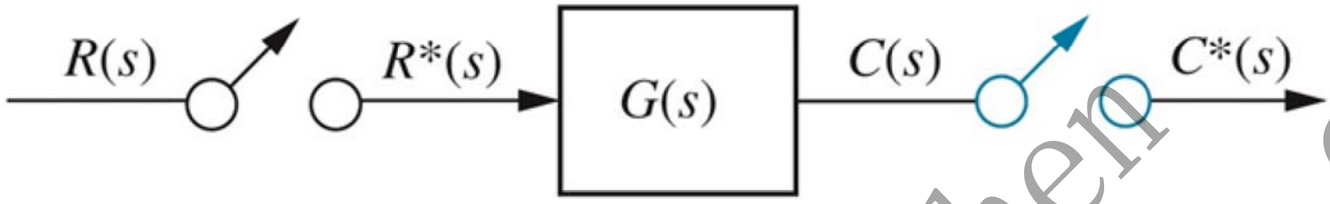
(c)

Note: Phantom sampler is shown in color.

If we are satisfied with finding the output at the sampling instants and not in between, the representation of the sampled-data system can be greatly simplified.

Using the concept described in Figure (c), we derive the ***PULSE TRANSFER FUNCTION*** of G(s).

Derivation of the Pulse Transfer Function



(c)

Note: Phantom sampler is shown in color.

$$f^*(t)=\sum_{n=0}^{\infty} f(nT)\delta(t-nT) \Rightarrow r^*(t)=\sum_{n=0}^{\infty} r(nT)\delta(t-nT)$$ which is a sum of impulses

Since the impulse response of a system, *G(s)*, is *g(t)*, we can write the time output of *G(s)* as the sum of impulse responses generated by the input:

$$c(t)=\sum_{n=0}^{\infty} r(nT)g(t-nT) \Rightarrow c(kT)=\sum_{n=0}^{\infty} r(nT)g(kT-nT)$$

$$F(z)=\sum_{k=0}^{\infty} f(kT)z^{-k} \Rightarrow C(z)=\sum_{k=0}^{\infty} c(kT)z^{-k}$$

$$\Rightarrow C(z)=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty} r(nT)g[(k-n)T]z^{-k}$$

$$C(z)=\sum_{k=0}^{\infty}\sum_{n=0}^{\infty}r(nT)g[(k-n)T]z^{-k}$$

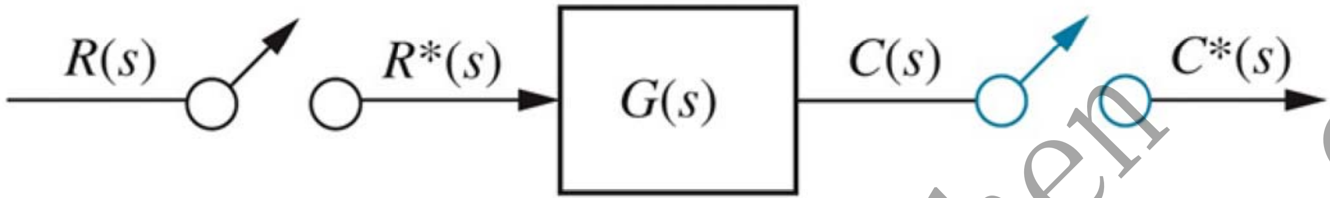
Letting $m=k-n$, we find

$$C(z)=\sum_{m+n=0}^{\infty}\sum_{n=0}^{\infty}r(nT)g[mT]z^{-(m+n)}$$
$$=\left\{\sum_{m=0}^{\infty}g(mT)z^{-m}\right\}\left\{\sum_{n=0}^{\infty}r(nT)z^{-n}\right\}$$

where the lower limit, $m+n$, was changed to m. The reasoning is that $m+n=0$ yields negative values of m for all $n > 0$. But, since $g(mT)=0$ for all $m < 0$, m is not less than zero. Alternately, $g(t)=0$ for t < 0. Thus, $n=0$ in the first sum's lower limit.

Using the definition of the z-transform

$$C(z)=\left\{\sum_{m=0}^{\infty}g(mT)z^{-m}\right\}\left\{\sum_{n=0}^{\infty}r(nT)z^{-n}\right\}=G(z)R(z)$$



(c)

Note: Phantom sampler is shown in color.

$$C(z)=\left\{\sum_{m=0}^{\infty} g(mT)z^{-m}\right\}\left\{\sum_{n=0}^{\infty} r(nT)z^{-n}\right\}=G(z)R(z)$$

This equation is a very important result, since it shows that the transform of the sampled output is the product of the transforms of the sampled input and the pulse transfer function of the system.
Remember that although the output of the system is a continuous function, we had to make an assumption of a sampled output (phantom sampler) in order to arrive at the compact result of this equation.

One way of finding the pulse transfer function, *G(z)*, is to start with *G(s)*, find *g(t)*, and then use Table 13.1 to find *G(z)*.

Example 13.4

Converting $G_1(s)$ in Cascade with z.o.h. to $G(z)$

PROBLEM: Given a z.o.h. in cascade with $G_1(s) = (s+2)/(s+1)$ or

$$G(s) = \frac{1-e^{-Ts}}{s}\frac{(s+2)}{(s+1)} \tag{13.38}$$

find the sampled-data transfer function, $G(z)$, if the sampling time, T, is 0.5 second.

SOLUTION: Equation (13.38) represents a common occurrence in digital control systems, namely a transfer function in cascade with a zero-order hold. Specifically, $G_1(s) = (s+2)/(s+1)$ is in cascade with a zero-order hold, $(1-e^{-Ts})/s$. We can formulate a general solution to this type of problem by moving the s in the denominator of the zero-order hold to $G_1(s)$, yielding

$$G(s) = (1-e^{-Ts})\frac{G_1(s)}{s} \tag{13.39}$$

from which

$$G(z) = (1-z^{-1})z\left\{\frac{G_1(s)}{s}\right\} = \frac{z-1}{z}z\left\{\frac{G_1(s)}{s}\right\} \tag{13.40}$$

Thus, begin the solution by finding the impulse response (inverse Laplace transform) of $G_1(s)/s$. Hence,

$$G_2(s) = \frac{G_1(s)}{s} = \frac{s+2}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{2}{s} - \frac{1}{s+1} \tag{13.41}$$

Taking the inverse Laplace transform, we get

$$g_2(t) = 2 - e^{-t} \tag{13.42}$$

from which

$$g_2(kT) = 2 - e^{-kt} \tag{13.43}$$

Using Table 13.1, we find

$$G_2(z) = \frac{2z}{z-1} - \frac{z}{z-e^{-T}} \tag{13.44}$$

Substituting $T = 0.5$ yields

$$G_2(z) = z\left\{\frac{G_1(s)}{s}\right\} = \frac{2z}{z-1} - \frac{z}{z-0.607} = \frac{z^2 - 0.213z}{(z-1)(z-0.607)} \tag{13.45}$$

From Eq. (13.40),

$$G(z) = \frac{z-1}{z} G_2(z) = \frac{z-0.213}{z-0.607} \tag{13.46}$$

Skill-Assessment Exercise 13.3

TryIt 13.2

Use MATLAB, the Control System Toolbox, and the following statements to solve Skill-Assessment Exercise 13.3.

```
Gs=zpk([],-4,8)
Gz=c2d(Gs,0.25,'zoh')
```

PROBLEM: Find $G(z)$ for $G(s) = 8/(s+4)$ in cascade with a zero-order sample and hold. The sampling period is 0.25 second.

ANSWER: $G(z) = 1.264/(z - 0.3679)$

The complete solution is located at www.wiley.com/college/nise.

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Since $G(s) = (1 - e^{-Ts})\dfrac{8}{s(s+4)}$,

$$G(z) = (1 - z^{-1})z\left\{\frac{8}{s(s+4)}\right\} = \frac{z-1}{z}z\left\{\frac{A}{s} + \frac{B}{s+4}\right\} = \frac{z-1}{z}z\left\{\frac{2}{s} + \frac{2}{s+4}\right\}.$$

Let $G_2(s) = \dfrac{2}{s} + \dfrac{2}{s+4}$. Therefore, $g_2(t) = 2 - 2e^{-4t}$, or $g_2(kT) = 2 - 2e^{-4kT}$.

Hence, $G_2(z) = \dfrac{2z}{z-1} - \dfrac{2z}{z - \mathrm{e}^{-4T}} = \dfrac{2z(1 - \mathrm{e}^{-4T})}{(z-1)(z - \mathrm{e}^{-4T})}$.

Therefore, $G(z) = \dfrac{z-1}{z}G_2(z) = \dfrac{2(1 - e^{-4T})}{(z - e^{-4T})}$.

For $T = \dfrac{1}{4}s$, $G(z) = \dfrac{1.264}{z - 0.3679}$

5. Block Diagram Reduction

- We have defined the z-transform and the sampled-data system transfer function and have shown how to obtain the sampled response.
- We now draw a parallel with some of the objectives of Chapter 5, namely block diagram reduction.
- Our objective here is to be able to find the closed-loop sampled-data transfer function of an arrangement of subsystems that have a computer in the loop.

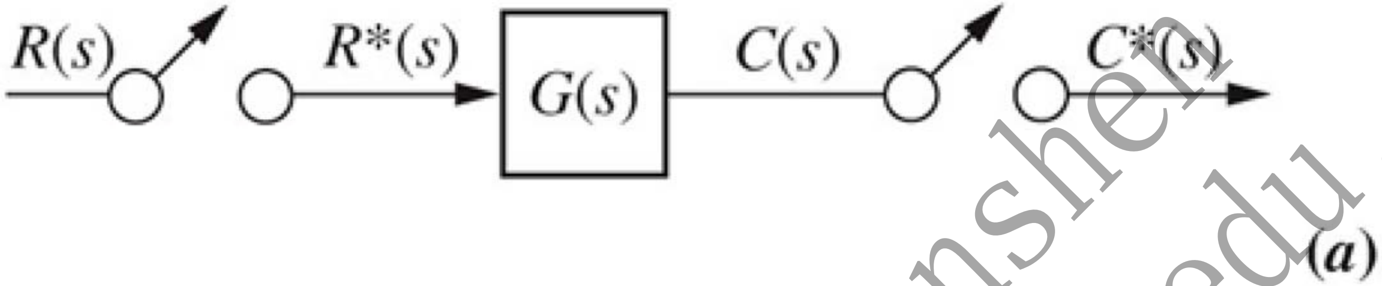
- When manipulating block diagrams for sampled-data systems, you must be careful to remember the definition of the sampled-data system transfer to avoid mistakes.

$$z\{G_1(s)G_2(s)\} \overset{?}{=} G_1(z)G_2(z)$$

$$z\{G_1(s)G_2(s)\} \neq G_1(z)G_2(z)$$

- The s-domain functions have to be multiplied together before taking the z-transform.
- We use the notation $G_1G_2(s)$ to denote a single function that is $G_1(s)G_2(s)$ after evaluating the product.

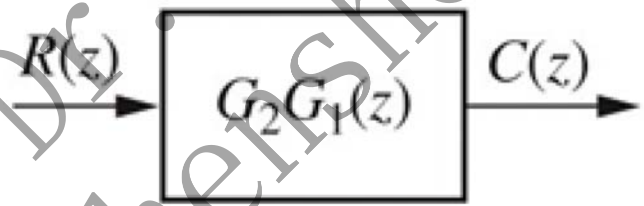
$$z\{G_1(s)G_2(s)\} = z\{G_1G_2(s)\} = G_1G_2(z) \neq G_1(z)G_2(z)$$

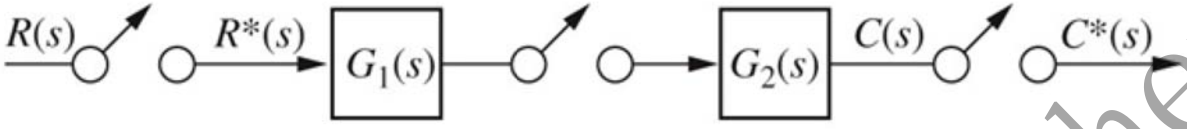


The standard system that we derived earlier is shown in Figure (a), where the transform of the output, C(z), is equal to R(z)G(z). This system forms the basis for the others.

(b)

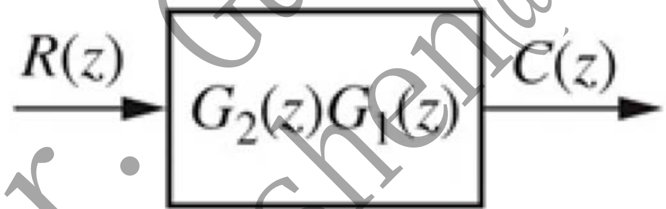
In Figure (b), there is no sampler between $G_1(s)$ and $G_2(s)$. Thus, we can think of a single function, $G_1(s)G_2(s)$, denoted $G_1G_2(s)$, existing between the two samplers and yielding a single transfer function, as shown in Figure (a). Hence, the pulse transfer function is $z\{G_1G_2(s)\}=G_1G_2(z)$. The transform of the output, $C(z)=R(z)G_1G_2(z)$.

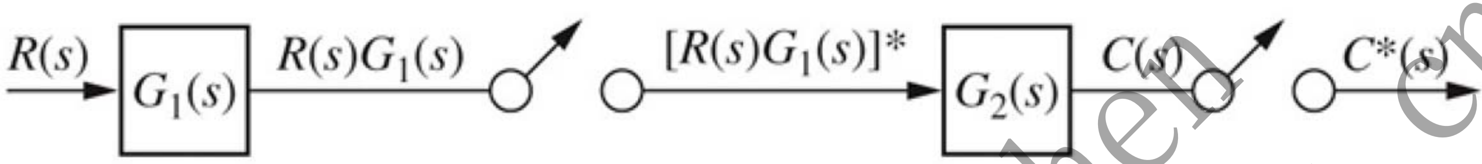




(c)

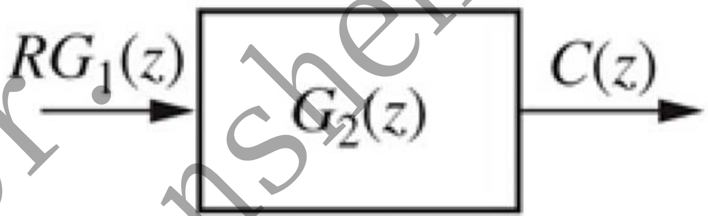
In Figure (c), we have the cascaded two subsystems of the type shown in Figure (a). For this case, then, the z-transform is the product of the two z-transforms, or $G_2(z)G_1(z)$. Hence the transform of the output $C(z)=R(z)\ G_2(z)G_1(z)$.





(d)

Finally, in Figure (d), we see that the continuous signal entering the sampler is $R(s)G_1(s)$. Thus, the model is the same as Figure (a) with $R(s)$ replaced by $R(s)G_1(s)$, and $G(s)$ in Figure (d) replacing $G(s)$ in Figure (a). The z-transform of the input to $G(s)$ is $z\{R(s)G_1(s)\}=z\{RG_1(s)\}=RG_1(z)$. The pulse transfer function for the system $G_2(s)$ is $G_2(z)$. Hence, the output $C(z)=RG_1(z)G_2(z)$.



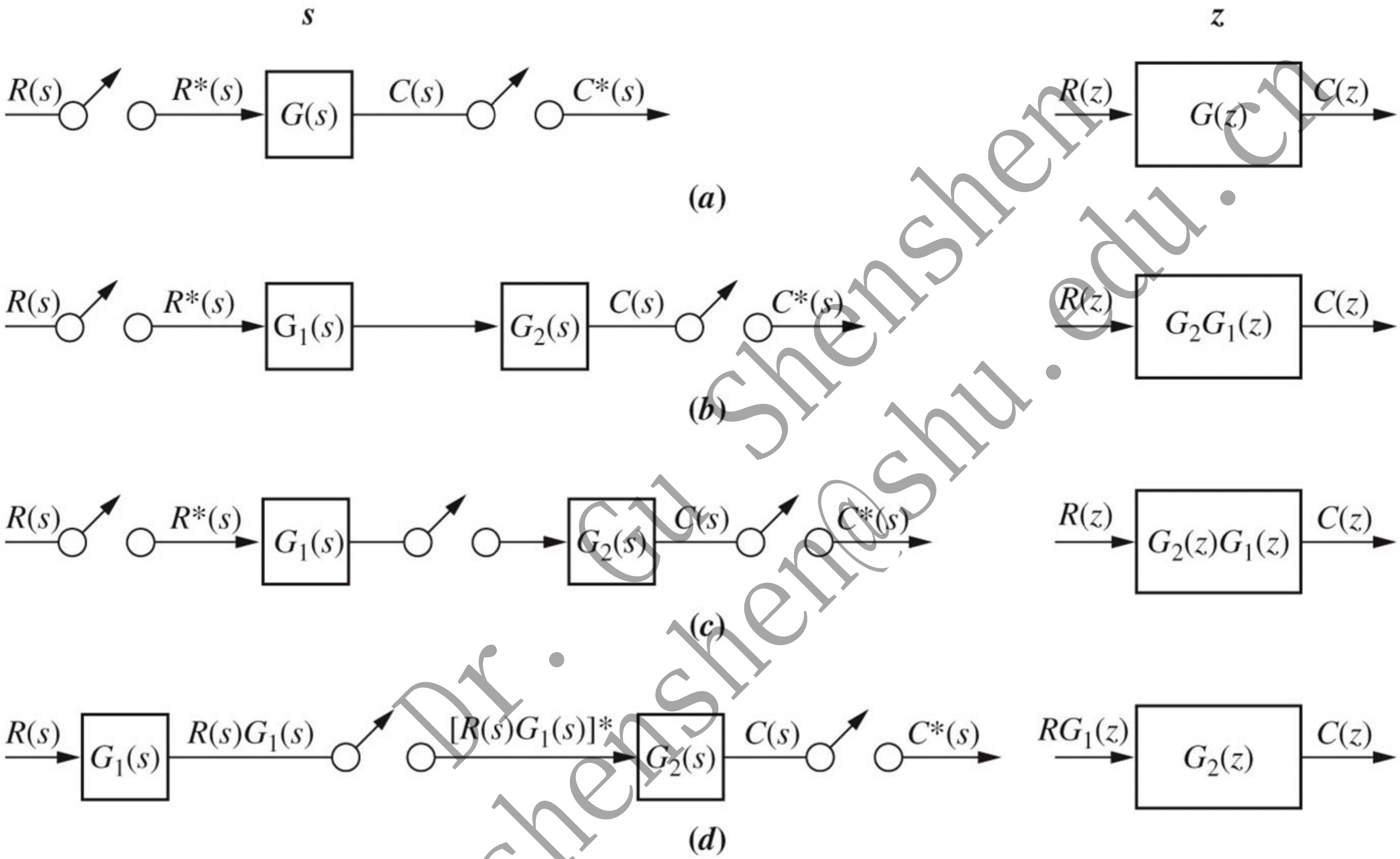

s
z
R(s)
R*(s)
G(s)
C(s)
C*(s)
R(z)
G(z)
C(z)
(a)
R(s)
R*(s)
G1(s)
G2(s)
C(s)
C*(s)
R(z)
G2G1(z)
C(z)
(b)
R(s)
R*(s)
G1(s)
G2(s)
C(s)
C*(s)
R(z)
G2(z)G1(z)
C(z)
(c)
R(s)
G1(s)
R(s)G1(s)
[R(s)G1(s)]*
G2(s)
C(s)
C*(s)
RG1(z)
G2(z)
C(z)
(d)

Figure 13.9

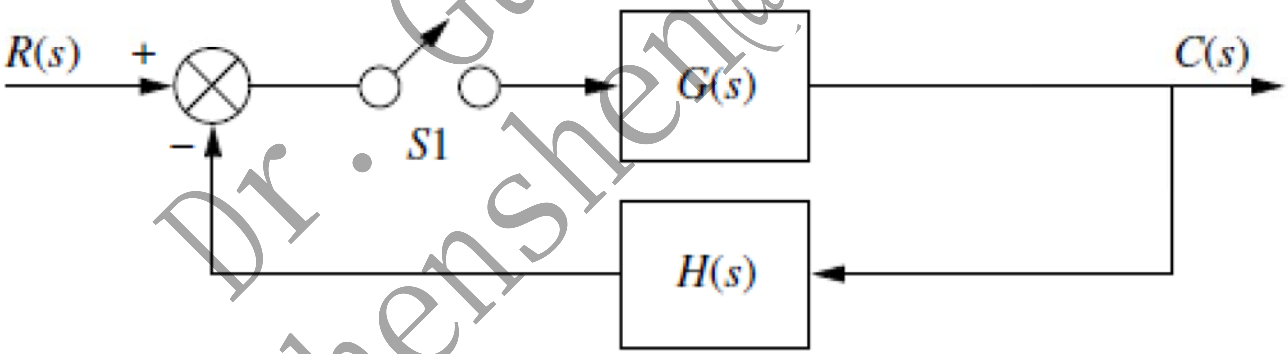
- Using the basic forms, we can now find the z-transform of feedback control systems.
- We have shown that any system, *G(s)*, with sampled input and sampled output, such as that shown in Figure (a), can be represented as a sampled-data transfer function, *G(z)*.
- Thus, we want to perform block diagram manipulations that result in subsystems, as well as the entire feedback system, that have sampled inputs and sampled outputs.
- Then we can make the transformation to sampled-data transfer functions.

Example 13.5

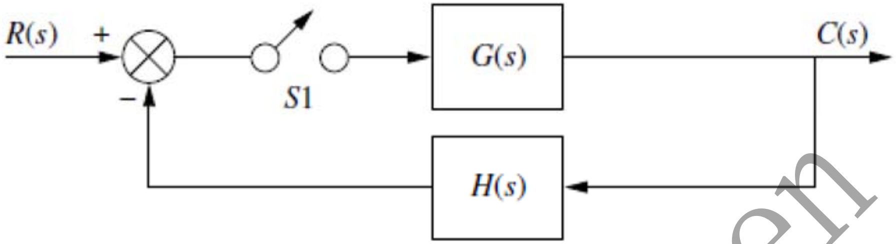
Pulse Transfer Function of a Feedback System

PROBLEM: Find the z-transform of the system shown in Figure 13.10(a).

SOLUTION: The objective of the problem is to proceed in an orderly fashion, starting with the block diagram of Figure 13.10(a) and reducing it to the one shown in Figure 13.10(f).



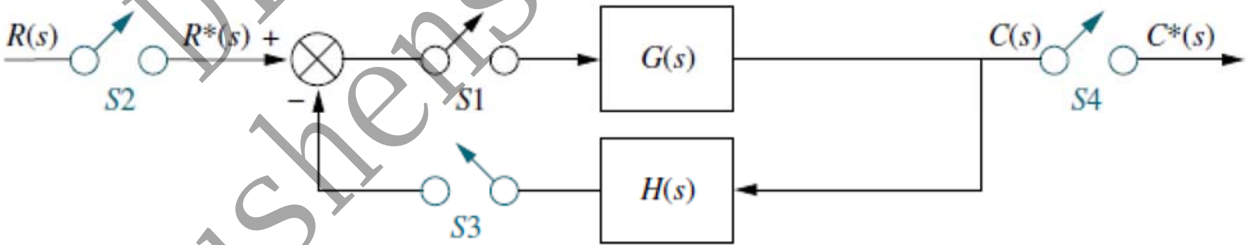
(a)



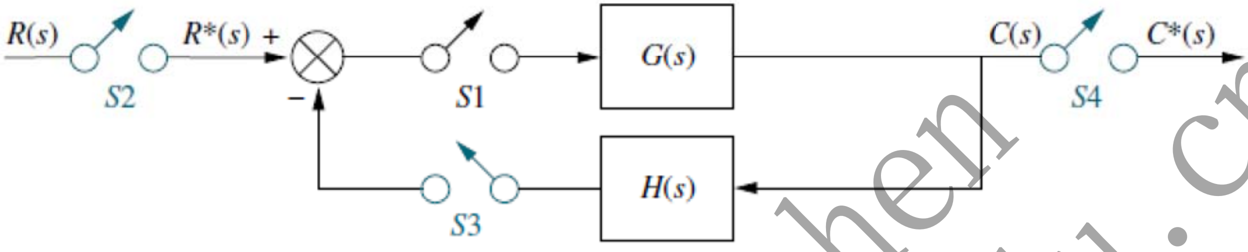
(a)

One operation we can always perform is to place a phantom sampler at the output of any subsystem that has a sampled input, provided that the nature of the signal sent to any other subsystem is not changed. For example in Figure 13.10(b), phantom sampler S4 can be added. The justification for this, of course, is that the output of a sampled-data system can only be found at the sampling instants anyway, and the signal is not an input to any other block.

Another operation that can be performed is to add phantom samplers S2 and S3 at the input to a summing junction whose output is sampled. The justification for this operation is that the sampled sum is equivalent to the sum of the sampled inputs, provided, of course, that all samplers are synchronized.

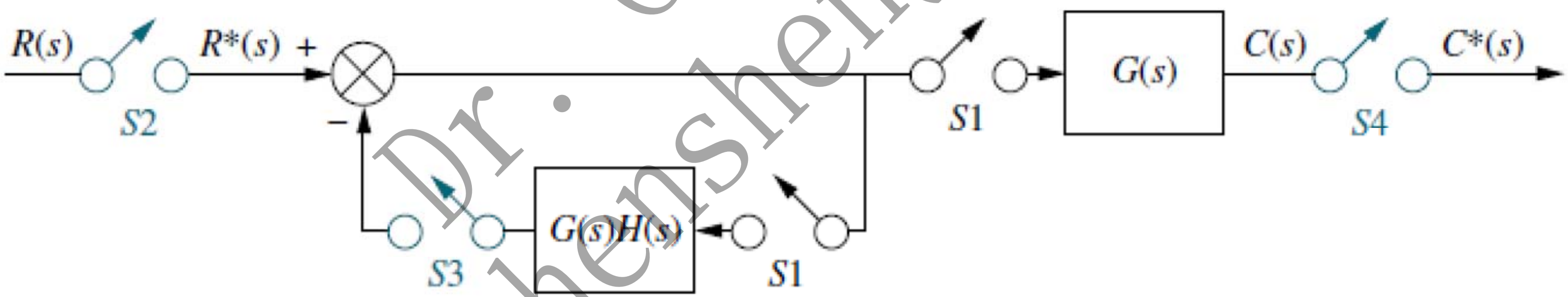


(b)

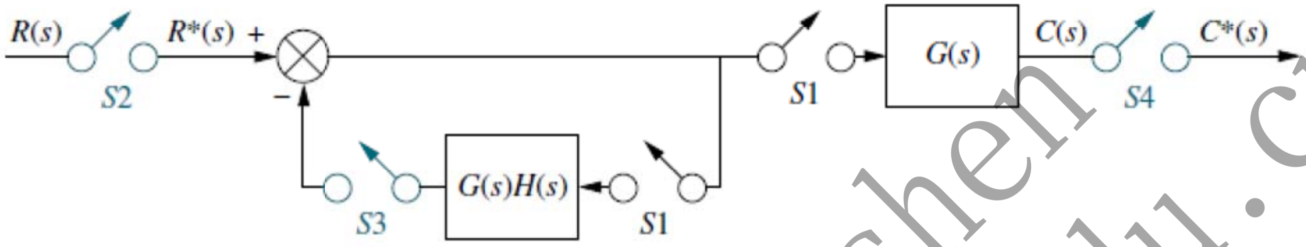


(b)

Next, move sampler $S1$ and $G(s)$ to the right past the pickoff point, as shown in Figure 13.10(c). The motivation for this move is to yield a sampler at the input of $G(s)H(s)$ to match Figure 13.9(b). Also, $G(s)$ with sampler $S1$ at the input and sampler $S4$ at the output matches Figure 13.9(a). The closed-loop system now has a sampled input and a sampled output.

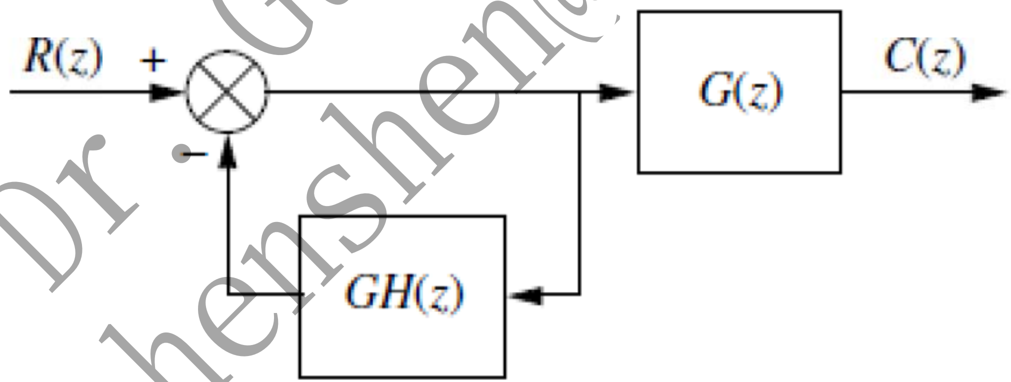


(c)

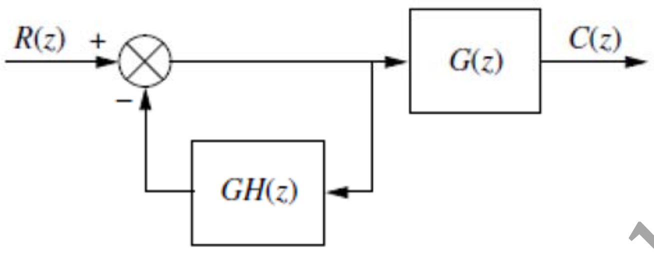


(c)

$G(s)H(s)$ with samplers $S1$ and $S3$ becomes $GH(z)$, and $G(s)$ with samplers $S1$ and $S4$ becomes $G(z)$, as shown in Figure 13.10(d). Also, converting $R^*(s)$ to $R(z)$ and $C^*(s)$ to $C(z)$, we now have the system represented totally in the z-domain.

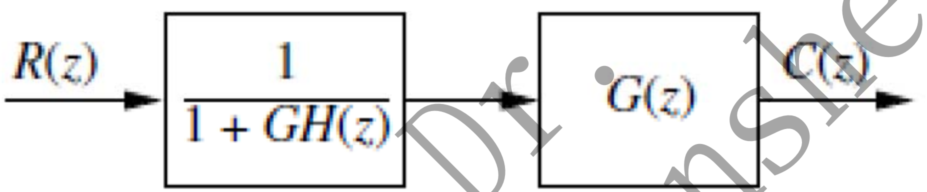


(d)

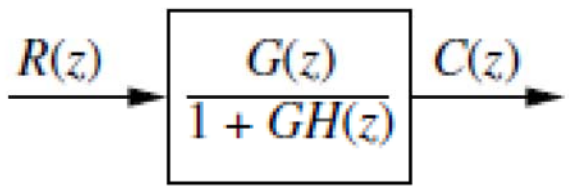


(*d*)

The equations derived in Chapter 5 for transfer functions represented with the Laplace transform can be used for sampled-data transfer functions with only a change in variables from s to z. Thus, using the feedback formula, we obtain the first block of Figure 13.10(e). Finally, multiplication of the cascaded sampled-data systems yields the final result shown in Figure 13.10(f).



(*e*)



(*f*)

Skill-Assessment Exercise 13.4

PROBLEM: Find $T(z) = C(z)/R(z)$ for the system shown in Figure 13.11.

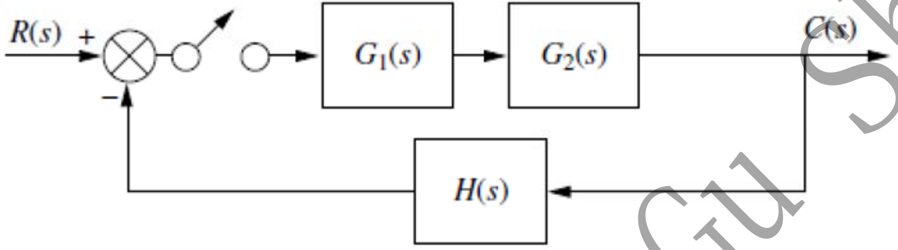


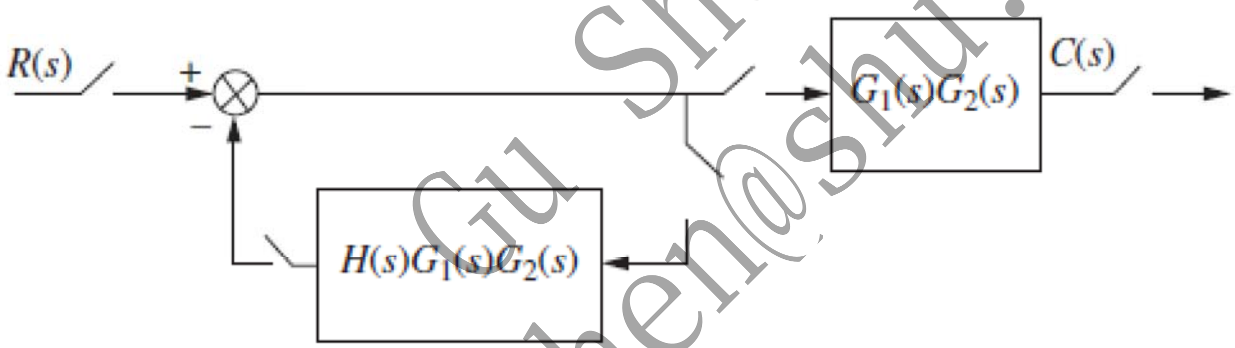
FIGURE 13.11 Digital system for Skill-Assessment Exercise 13.4

ANSWER: $T(z) = \dfrac{G_1G_2(z)}{1 + HG_1G_2(z)}$

The complete solution is located at www.wiley.com/college/nise.

13.4

Add phantom samplers to the input, feedback after $H(s)$, and to the output. Push $G_1(s)G_2(s)$, along with its input sampler, to the right past the pickoff point and obtain the block diagram shown below.



Hence, $T(z) = \dfrac{G_1G_2(z)}{1 + HG_1G_2(z)}$.

6. Stability

- The glaring difference between analog feedback control systems and digital feedback control systems is the effect that the sampling rate has on the transient response.
- Changes in sampling rate not only change the nature of the response from overdamped to underdamped, but also can turn a stable system into an unstable one.
- We now discuss the stability of digital systems from two perspectives:
  - (1) z-plane and
  - (2) s-plane.
- We will see that the Routh-Hurwitz criterion can be used only if we perform our analysis and design on the s-plane.

Digital System Stability via the z-Plane

- In the s-plane, the region of stability is the left half-plane. If the transfer function, $G(s)$, is transformed into a sampled-data transfer function, $G(z)$, the region of stability on the z-plane can be evaluated from the definition, $z=e^{Ts}$.

- Letting $s=\alpha+j\omega$, we obtain

$$z = e^{Ts} = e^{T(\alpha+j\omega)} = e^{\alpha T} e^{j\omega T}$$

$$= e^{\alpha T}\left(\cos\omega T + j\sin\omega T\right)$$

$$= e^{\alpha T}\angle\omega T$$

$$\text{Since } \left(\cos\omega T + j\sin\omega T\right) = 1\angle\omega T$$

- Each region of the s-plane can be mapped into a corresponding region on the z-plane. Points that have positive values of α are in the right half of the s-plane, region C. The magnitudes of the mapped points are $e^{\alpha T} > 1$. Thus points in the right half of the s-plane map into points outside the unit circle on the z-plane.

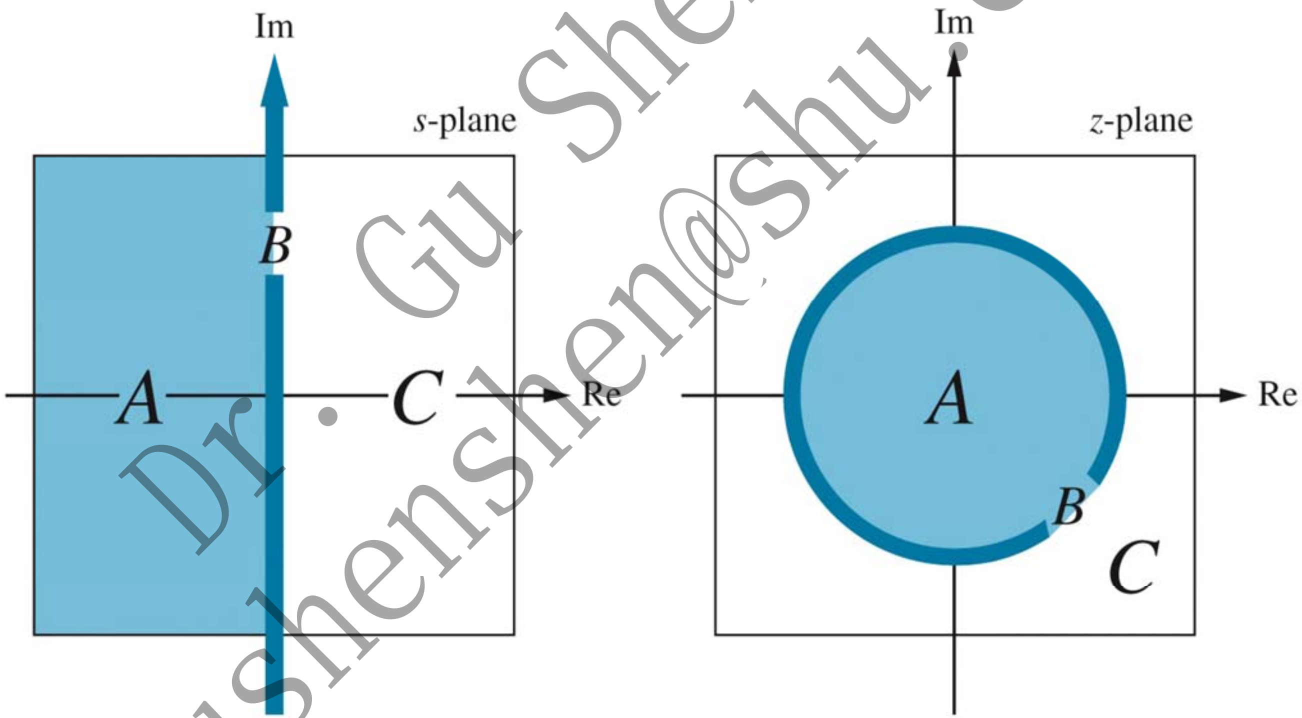


Figure 13.13

- Points on the $j\omega$-axis, region B, have zero values of α and yield points on the z-plane with magnitude=1, the unit circle. Hence, points on the $j\omega$-axis in the s-plane map into points on the unit circle on the z-plane.

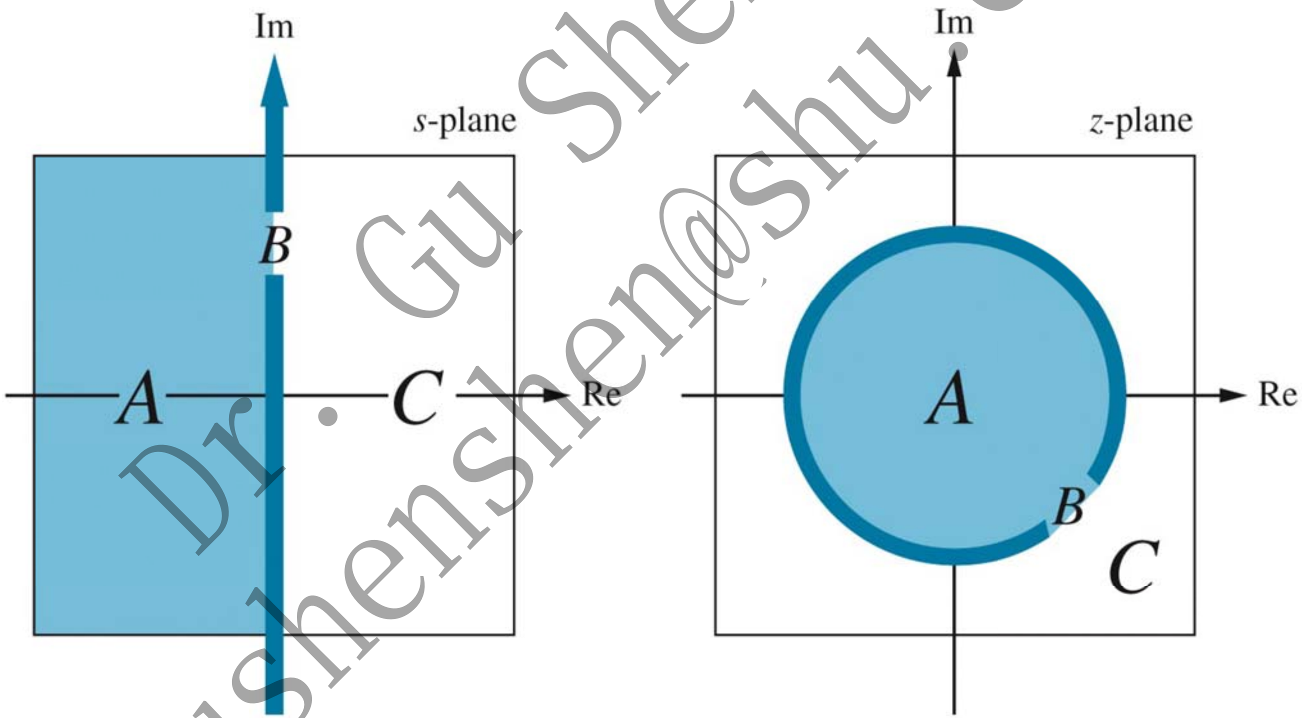


Figure 13.13

- Finally, points on the s-plane that yield negative values of a α(left–half-plane roots, region A) map into the inside of the unit circle on the z-plane.

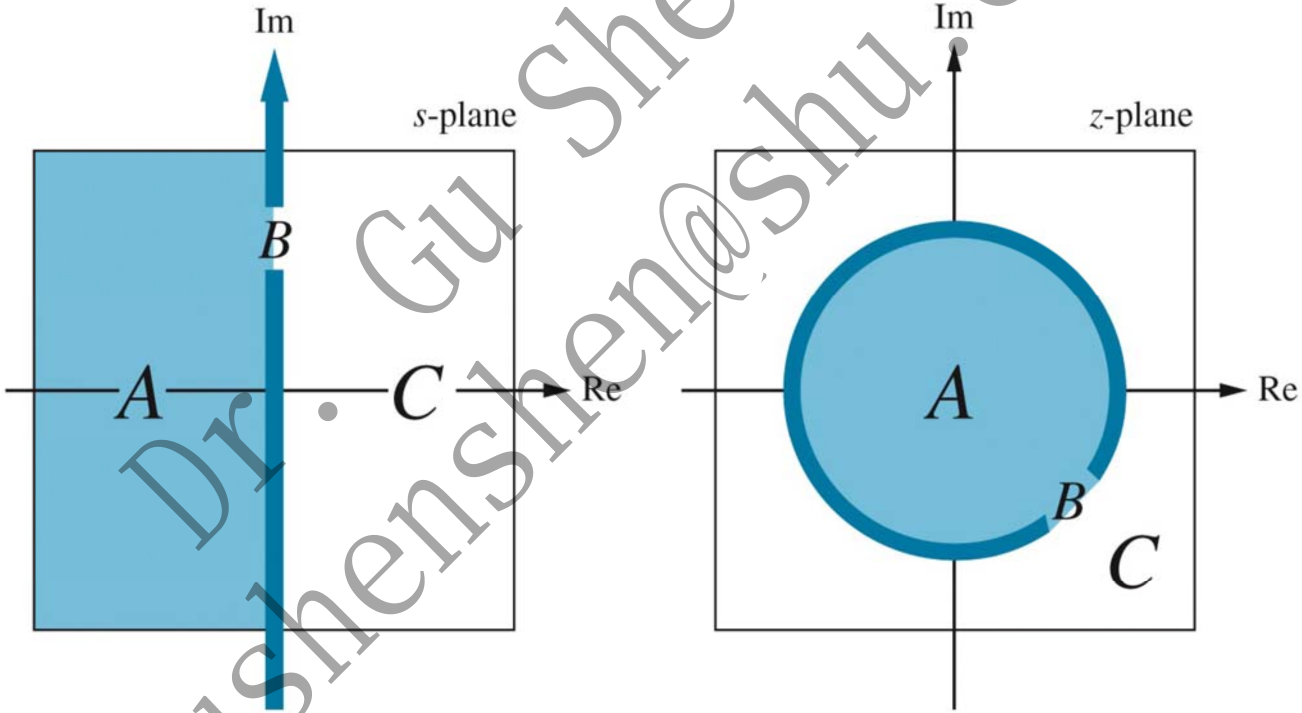


Figure 13.13

- Thus, a digital control system is
  - (1) stable if all poles of the closed-loop transfer function, T(z), are inside the unit circle on the z-plane,
  - (2) unstable if any pole is outside the unit circle and/or there are poles of multiplicity greater than one on the unit circle, and
  - (3) marginally stable if poles of multiplicity one are on the unit circle and all other poles are inside the unit circle.

Example 13.7

Range of T for Stability

PROBLEM: Determine the range of sampling interval, T, that will make the system shown in Figure 13.15 stable, and the range that will make it unstable.

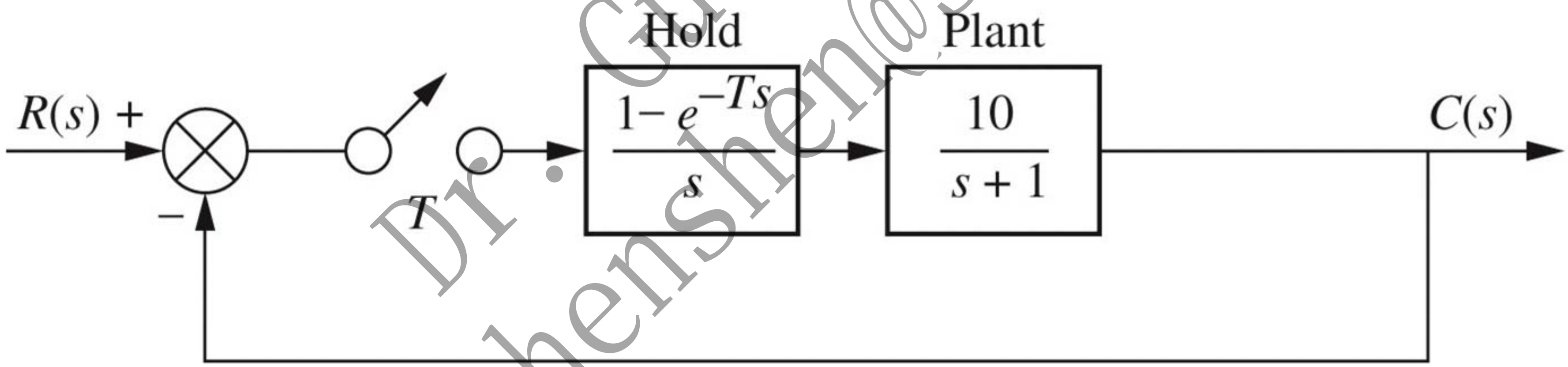
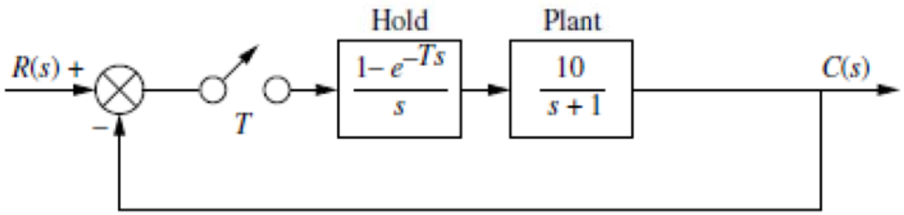


Figure 13.15

SOLUTION: Since $H(s) = 1$, the z-transform of the closed-loop system, $T(z)$, is found from Figure 13.10 to be

$$T(z) = \frac{G(z)}{1 + G(z)} \tag{13.53}$$



To find $G(z)$, first find the partial-fraction expansion of $G(s)$.

$$G(s) = 10\frac{1 - e^{-Ts}}{s(s+1)} = 10(1 - e^{-Ts})\left(\frac{1}{s} - \frac{1}{s+1}\right) \tag{13.54}$$

Taking the z-transform, we obtain

$$G(z) = \frac{10(z-1)}{z}\left[\frac{z}{z-1} - \frac{z}{z - e^{-T}}\right] = 10\frac{(1 - e^{-T})}{(z - e^{-T})} \tag{13.55}$$

Substituting Eq. (13.55) into (13.53) yields

$$T(z) = \frac{10(1 - e^{-T})}{z - (11e^{-T} - 10)} \tag{13.56}$$

The pole of Eq. (13.56), $(11e^{-T} - 10)$, monotonically decreases from $+1$ to -1 for $0 < T < 0.2$. For $0.2 < T < \infty$, $(11e^{-T} - 10)$ monotonically decreases from -1 to -10. Thus, the pole of $T(z)$ will be inside the unit circle, and the system will be stable if $0 < T < 0.2$. In terms of frequency, where $f = 1/T$, the system will be stable as long as the sampling frequency is $1/0.2 = 5$ hertz or greater.

- We now have found, via the z-plane, that sampled systems are stable if their poles are inside the unit circle.
- In the case of continuous systems, the determination of stability hinges upon our ability to determine whether the roots of the denominator of the closed-loop transfer function are in the stable region of the s-plane.
- The problem for high-order systems is complicated by the fact that the closed-loop transfer function denominator is in polynomial form, not factored form.
- The same problem surfaces with closed loop sampled-data transfer functions.

Discuss

- Can we apply Routh-Hurwitz criterion to determine the stability of a sampled system?
- It depends on how we apply the Routh-Hurwitz criterion?
- Bad news: This stability criterion precludes the use of the Routh-Hurwitz criterion, which detects roots in the right half-plane rather than outside the unit circle.
- Good news: Another method exists that allows us to use the familiar s-plane and the Routh-Hurwitz criterion to determine the stability of a sampled system.

Bilinear Transformations

- Bilinear transformations give us the ability to apply our s-plane analysis and design techniques to digital systems.
- We can analyze and design on the s-plane and then, using these transformations, convert the results to a digital system that contains the same properties.

$$G(z) \xrightarrow{z=e^{Ts}} G\left(e^{Ts}\right)$$

$$G(s) \xrightarrow{s=(1/T)\ln z} G\left((1/T)\ln z\right)$$

Both transformations yield *transcendental functions*. **They are too complicated to solve.**

- What we would like is a simple transformation that would yield linear arguments when transforming in both directions (bilinear) through direct substitution and without the complicated z-transform.

- Bilinear transformations of the form $z = \dfrac{as+b}{cs+d}$

And its inverse, $s = \dfrac{-dz+b}{cz-a}$

- Different values of a, b, c, and d have been derived for particular applications and yield various degrees of accuracy when comparing properties of the continuous and sampled functions.
- A particular choice of coefficients will take points on the unit circle and map them into points on the $j\omega$-axis. Points outside the unit circle will be mapped into the right half-plane, and points inside the unit circle will be mapped into the left half-plane. Thus, we will be able to make a simple transformation from the *z*-plane to the *s*-plane and obtain stability information about the digital system by working in the *s*-plane.
- Since the transformations are not exact, only the property for which they are designed can be relied upon. For the stability transformation just discussed, we cannot expect the resulting *G(s)* to have the same transient response as *G(z)*.

Digital System Stability via the s-Plane

- We look at a bilinear transformation that maps jω-axis points on the s-plane to unit-circle points on the z-plane.
- Further, the transformation maps right–half-plane points on the s-plane to points outside the unit circle on the z-plane.
- Finally, the transformation maps left–half-plane points on the s-plane to points inside the unit circle on the z-plane.
- Thus, we are able to transform the denominator of the pulsed transfer function, *D(z)*, to the denominator of a continuous transfer function, *D(s)*, and use the Routh-Hurwitz criterion to determine stability.
- Bilinear transformation $s = \frac{z+1}{z-1}$

and its inverse, $z = \frac{s+1}{s-1}$

perform the required transformation.

$$z = \frac{s+1}{s-1} \xrightarrow{s=\alpha+j\omega} z = \frac{(\alpha+1)+j\omega}{(\alpha-1)+j\omega}$$

$$|z| = \frac{\sqrt{(\alpha+1)^2+\omega^2}}{\sqrt{(\alpha-1)^2+\omega^2}} \Rightarrow \begin{cases} |z|<1 & \text{when } \alpha<0 \\ |z|>1 & \text{when } \alpha>0 \\ |z|=1 & \text{when } \alpha=0 \end{cases}$$

Example 13.8

Stability via Routh-Hurwitz

PROBLEM: Given $T(z) = N(z)/D(z)$, where $D(z) = z^3 - z^2 - 0.2z + 0.1$, use the Routh-Hurwitz criterion to find the number of z-plane poles of $T(z)$ inside, outside, and on the unit circle. Is the system stable?

SOLUTION: Substitute Eq. (13.60) into $D(z) = 0$ and obtain[3]

$$s^3 - 19s^2 - 45s - 17 = 0 \tag{13.64}$$

The Routh table for Eq. (13.64), Table 13.3, shows one root in the right–half-plane and two roots in the left–half-plane. Hence, $T(z)$ has one pole outside the unit circle, no poles on the unit circle, and two poles inside the unit circle. The system is unstable because of the pole outside the unit circle.

TABLE 13.3 Routh table for Example 13.8

s^3	1	–45
s^2	19	–17
s^1	–45.89	0
s^0	–17	0

Skill-Assessment Exercise 13.5

WileyPLUS
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Control Solutions

PROBLEM: Determine the range of sampling interval, T, that will make the system shown in Figure 13.16 stable.

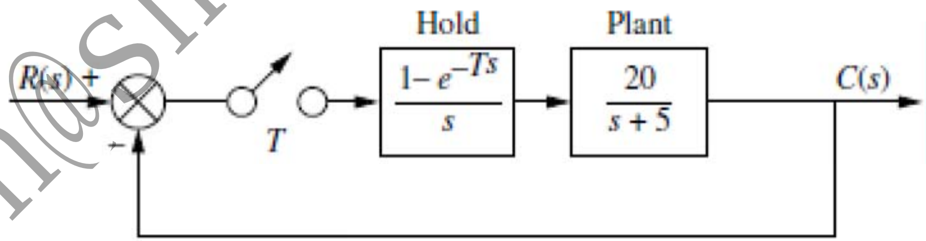


FIGURE 13.16 Digital system for Skill-Assessment Exercise 13.5

ANSWER: $0 < T < 0.1022$ second

The complete solution is located at www.wiley.com/college/nise.

13.5

Let $G(s)=\frac{20}{s+5}$. Let $G_2(s)=\frac{G(s)}{s}=\frac{20}{s(s+5)}=\frac{4}{s}-\frac{4}{s+5}$. Taking the inverse Laplace transform and letting $t=kT$, $g_2(kT)=4-4e^{-5kT}$. Taking the z-transform yields $G_2(z)=\frac{4z}{z-1}-\frac{4z}{z-e^{-5T}}=\frac{4z(1-e^{-5T})}{(z-1)(z-e^{-5T})}$.

Now, $G(z)=\frac{z-1}{z}G_2(z)=\frac{4(1-e^{-5T})}{(z-e^{-5T})}$.

Finally, $T(z)=\frac{G(z)}{1+G(z)}=\frac{4(1-e^{-5T})}{z-5e^{-5T}+4}$.

The pole of the closed-loop system is at $5e^{-5T}-4$. Substituting values of T, we find that the pole is greater than 1 if $T>0.1022$ s. Hence, the system is stable for $0<T<0.1022$ s.

Skill-Assessment Exercise 13.6

PROBLEM: Given $T(z) = N(z)/D(z)$, where $D(z) = z^3 - z^2 - 0.5z + 0.3$, use the Routh-Hurwitz criterion to find the number of z-plane poles of $T(z)$ inside, outside, and on the unit circle. Is the system stable?

ANSWER: $T(z)$ has one pole outside the unit circle, no poles on the unit circle, and two poles inside the unit circle. The system is unstable.

The complete solution is located at www.wiley.com/college/nise.

13.6

Substituting $z = \frac{s+1}{s-1}$ into $D(z) = z^3 - z^2 - 0.5z + 0.3$, we obtain $D(s) = s^3 - 8s^2 - 27s - 6$. The Routh table for this polynomial is shown below.

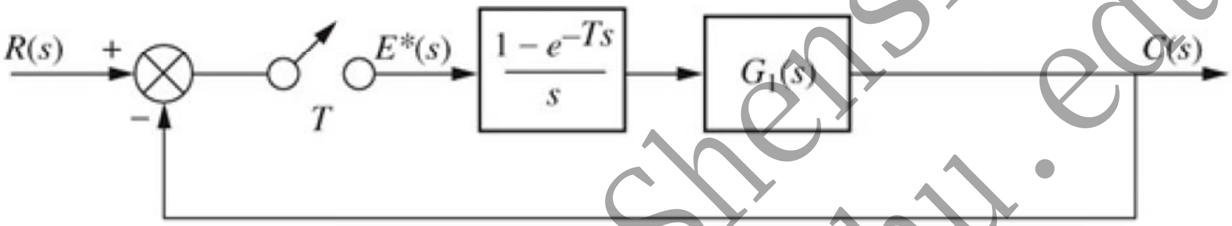
s^3	1	–27
s^2	–8	–6
s^1	–27.75	0
s^0	–6	0

Since there is one sign change, we conclude that the system has one pole outside the unit circle and two poles inside the unit circle. The table did not produce a row of zeros and thus, there are no $j\omega$ poles. The system is unstable because of the pole outside the unit circle.

7. Steady-State Errors

- We now examine the effect of sampling upon the steady-state error for digital systems.
- Any general conclusion about the steady-state error is difficult because of the dependence of those conclusions upon the placement of the sampler in the loop.
- Remember that the position of the sampler could change the open-loop transfer function.
- In the discussion of analog systems, there was only one open-loop transfer function, G(s), upon which the general theory of steady-state error was based and from which came the standard definitions of static error constants.
- For digital systems, however, the placement of the sampler changes the open-loop transfer function and thus precludes any general conclusions.
- In this section, we assume the typical placement of the sampler after the error and in the position of the cascade controller, and we derive our conclusions accordingly about the steady-state error of digital systems.

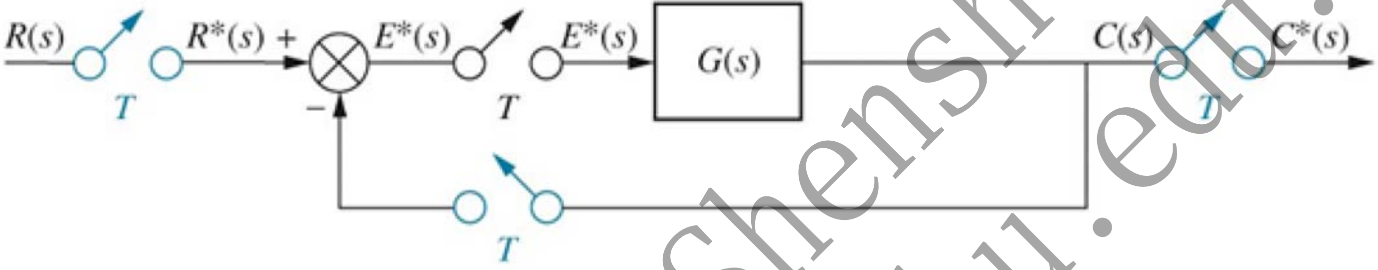
- Consider the digital system in Figure (a), where the digital computer is represented by the sampler and zero-order hold.



(a)

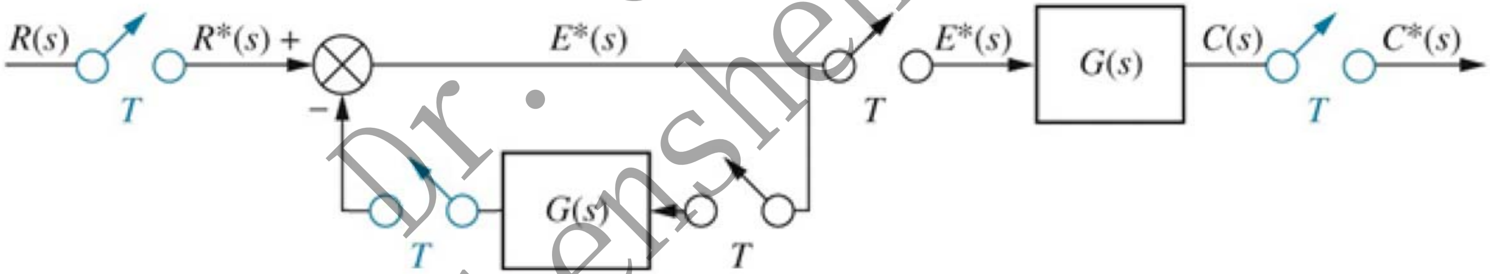
- The transfer function of the plant is represented by $G_1(s)$ and the transfer function of the z.o.h. by $(1-e^{-Ts})/s$. Letting $G(s)$ equal the product of the z.o.h. and $G_1(s)$, and using the block diagram reduction techniques for sampled-data systems, we can find the sampled error, $E^*(s)=E(z)$.

- Adding synchronous samplers at the input and the feedback, we obtain Figure (b).



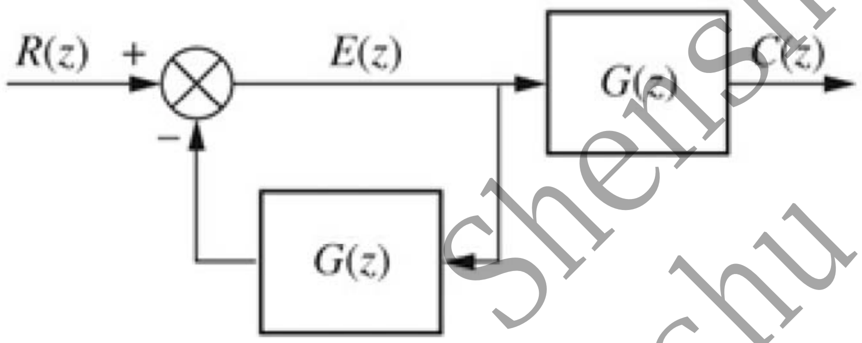
(b)

- Pushing *G(s)* and its input sampler to the right past the pickoff point yields Figure (c).



(c)

- Using Figure (a), we can convert each block to its z-transform, resulting in Figure (d).



(d)

- From this figure, E(z)=R(z)-E(z)G(z), or $E(z)=\dfrac{R(z)}{1+G(z)}$

- The final value theorem for discrete signals states that $e^{*}(\infty)=\lim_{z\to 1}\left(1-z^{-1}\right)E(z)$

where *e*(∞)* is the final sampled value of *e(t)*, or (alternatively) the final value of *e(kT)*.

- Using the final value theorem, the sampled steady state error, $e^*(\infty)$, for unity negative-feedback systems is

$$e^*(\infty) = \lim_{z \to 1}\left(1 - z^{-1}\right)E(z) = \lim_{z \to 1}\left(1 - z^{-1}\right)\frac{R(z)}{1 + G(z)}$$

- The above equation must now be evaluated for each input: step, ramp, and parabola.

Unit Step Input

- For a unit step input, $R(z)=\dfrac{z}{z-1}$

$$e^*(\infty)=\lim_{z\to 1}\left(1-z^{-1}\right)\frac{R(z)}{1+G(z)}\bigg|_{R(z)=\frac{z}{z-1}}=\frac{1}{1+\lim\limits_{z\to 1}G(z)}$$

- Defining the static error constant, K_p, as

$$K_p=\lim_{z\to 1}G(z)$$

$$e^*(\infty)=\frac{1}{1+K_p}$$

Unit Ramp Input

- For a unit ramp input, $R(z)=\frac{Tz}{(z-1)^2}$

$$e^*(\infty)=\frac{1}{K_v}$$

$$\text{where } K_v=\frac{1}{T}\lim_{z\to 1}(z-1)G(z)$$

Unit Parabolic Input

- For a unit parabolic input, $R(z)=\dfrac{T^2 z(z+1)}{2(z-1)^3}$

$$e^*(\infty)=\frac{1}{K_a}$$

$$\text{where } K_a=\frac{1}{T^2}\lim_{z\to 1}(z-1)^2 G(z)$$

Summary of Steady-State Errors

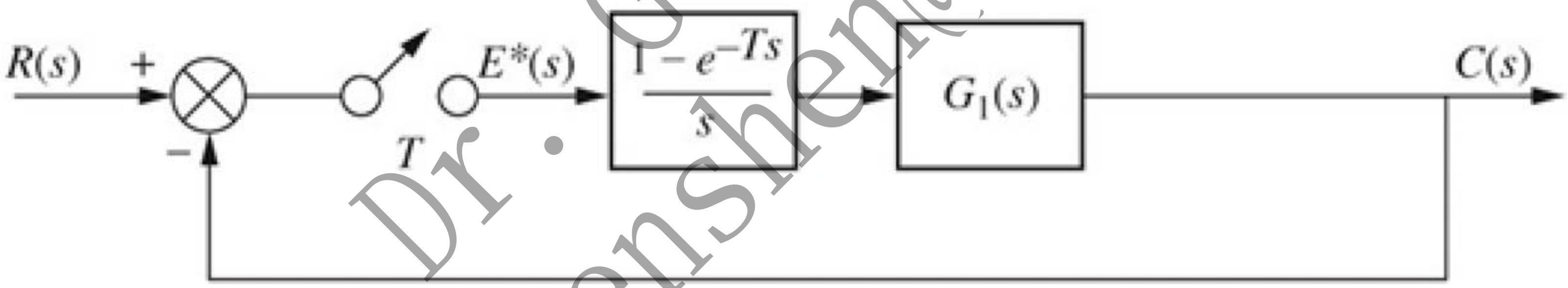
- The equations developed above for $e^*(\infty)$, K_p, K_v, and K_a are similar to the equations developed for analog systems.
- Multiple pole placement at the origin of the **s-plane** reduced steady-state errors to zero in the **analog case**.
- Multiple pole placement at $z=1$ reduces the steady-state error to zero for **digital systems** of the type discussed in this section.
- This conclusion makes sense when one considers that $s=0$ maps into $z=1$ under $z=e^{Ts}$.
  - For a step input, if $G(z)$ has one pole at $z=1$, the limit will become infinite, and the steady-state error will reduce to zero.
  - For a ramp input, if $G(z)$ has two poles at $z=1$, the limit will become infinite, and the error will reduce to zero.
  - Similar conclusions can be drawn for the parabolic input. Here, $G(z)$ needs three poles at $z=1$ in order for the steady-state error to be zero.

Example 13.9

Finding Steady-State Error

PROBLEM: For step, ramp, and parabolic inputs, find the steady-state error for the feedback control system shown in Figure 13.17(*a*) if

$$G_1(s) = \frac{10}{s(s+1)} \tag{13.76}$$



SOLUTION: First find $G(s)$, the product of the z.o.h. and the plant.

$$G(s)=\frac{10(1-e^{-Ts})}{s^2(s+1)}=10(1-e^{-Ts})\left[\frac{1}{s^2}-\frac{1}{s}+\frac{1}{s+1}\right] \tag{13.77}$$

The z-transform is then

$$G(z)=10(1-z^{-1})\left[\frac{Tz}{(z-1)^2}-\frac{z}{z-1}+\frac{z}{z-e^{-T}}\right]$$
$$=10\left[\frac{T}{z-1}-1+\frac{z-1}{z-e^{-T}}\right] \tag{13.78}$$

For a step input,

$$K_p=\lim_{z\to 1}G(z)=\infty;\quad e^*(\infty)=\frac{1}{1+K_p}=0 \tag{13.79}$$

For a ramp input,

$$K_v=\frac{1}{T}\lim_{z\to 1}(z-1)G(z)=10;\quad e^*(\infty)=\frac{1}{K_v}=0.1 \tag{13.80}$$

For a parabolic input,

$$K_a=\frac{1}{T^2}\lim_{z\to 1}(z-1)^2G(z)=0;\quad e^*(\infty)=\frac{1}{K_a}=\infty \tag{13.81}$$

You will notice that the answers obtained are the same as the results obtained for the analog system. However, since stability depends upon the sampling interval, be sure to check the stability of the system after a sampling interval is established before making steady-state error calculations.

Skill-Assessment Exercise 13.7

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Control Solutions

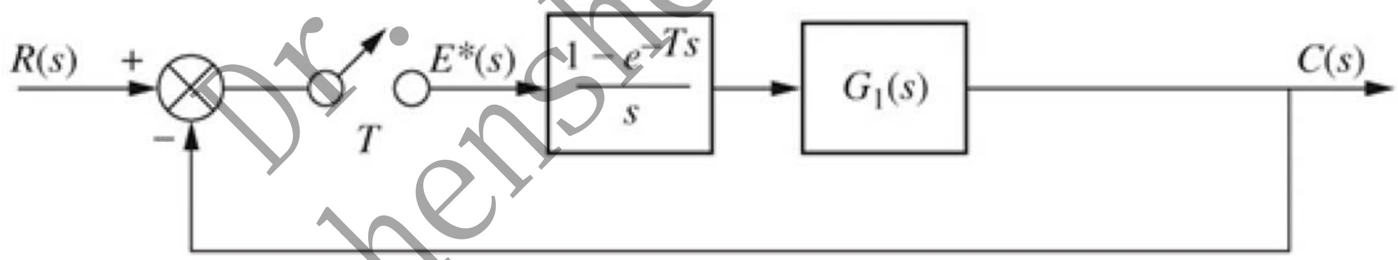
PROBLEM: For step, ramp, and parabolic inputs, find the steady-state error for the feedback control system shown in Figure 13.17(*a*) if

$$G_1(s) = \frac{20(s+3)}{(s+4)(s+5)}$$

Let $T = 0.1$ second. Repeat for $T = 0.5$ second.

ANSWER: For $T = 0.1$ second, $K_p = 3$, $K_v = 0$, and $K_a = 0$; for $T = 0.5$ second, the system is unstable.

The complete solution is located at www.wiley.com/college/nise.



13.7

Defining $G(s)$ as $G_1(s)$ in cascade with a zero-order-hold,

$$G(s) = 20(1 - e^{-Ts})\left[\frac{(s+3)}{s(s+4)(s+5)}\right] = 20(1 - e^{-Ts})\left[\frac{3/20}{s} + \frac{1/4}{(s+4)} - \frac{2/5}{(s+5)}\right].$$

Taking the z-transform yields

$$G(z) = 20(1 - z^{-1})\left[\frac{(3/20)z}{z-1} + \frac{(1/4)z}{z - e^{-4T}} - \frac{(2/5)z}{z - e^{-5T}}\right] = 3 + \frac{5(z-1)}{z - e^{-4T}} - \frac{8(z-1)}{z - e^{-5T}}.$$

Hence for $T = 0.1$ second, $K_p = \lim_{z \to 1} G(z) = 3$, and $K_v = \frac{1}{T} \lim_{z \to 1} (z-1)G(z) = 0$, and $K_a = \frac{1}{T^2} \lim_{z \to 1} (z-1)^2 G(z) = 0$. Checking for stability, we find that the system is stable for $T = 0.1$ second, since $T(z) = \frac{G(z)}{1+G(z)} = \frac{1.5z - 1.109}{z^2 + 0.222z - 0.703}$ has poles inside the unit circle at -0.957 and $+0.735$. Again, checking for stability, we find that the system is unstable for $T = 0.5$ second, since $T(z) = \frac{G(z)}{1+G(z)} = \frac{3.02z - 0.6383}{z^2 + 2.802z - 0.6272}$ has poles inside and outside the unit circle at $+0.208$ and -3.01, respectively.

8. Transient Response on the z-Plane

- For analog systems a transient response requirement was specified by selecting a closed-loop, s-plane pole.
- We established the relationships between transient response and the s-plane.
- We saw that vertical lines on the s-plane were lines of constant settling time, horizontal lines were lines of constant peak time, and radial lines were lines of constant percent overshoot.
- In order to draw equivalent conclusions on the z-plane, we now map those lines through $z=e^{sT}$.

Vertical lines

- The vertical lines on the s-plane are lines of constant settling time and are characterized by the equation $s=\sigma_1+j\omega$, where the real part, $\sigma_1=-4/T_s$, is constant and is in the left–half-plane for stability. Substituting this into $z=e^{sT}$, we obtain

$$z = e^{\sigma_1 T} e^{j\omega T} = r_1 e^{j\omega T}$$

- It denotes concentric circles of radius r_1
- If σ_1 is positive, the circle has a larger radius than the unit circle.
- On the other hand, if σ_1 is negative, the circle has a smaller radius than the unit circle.

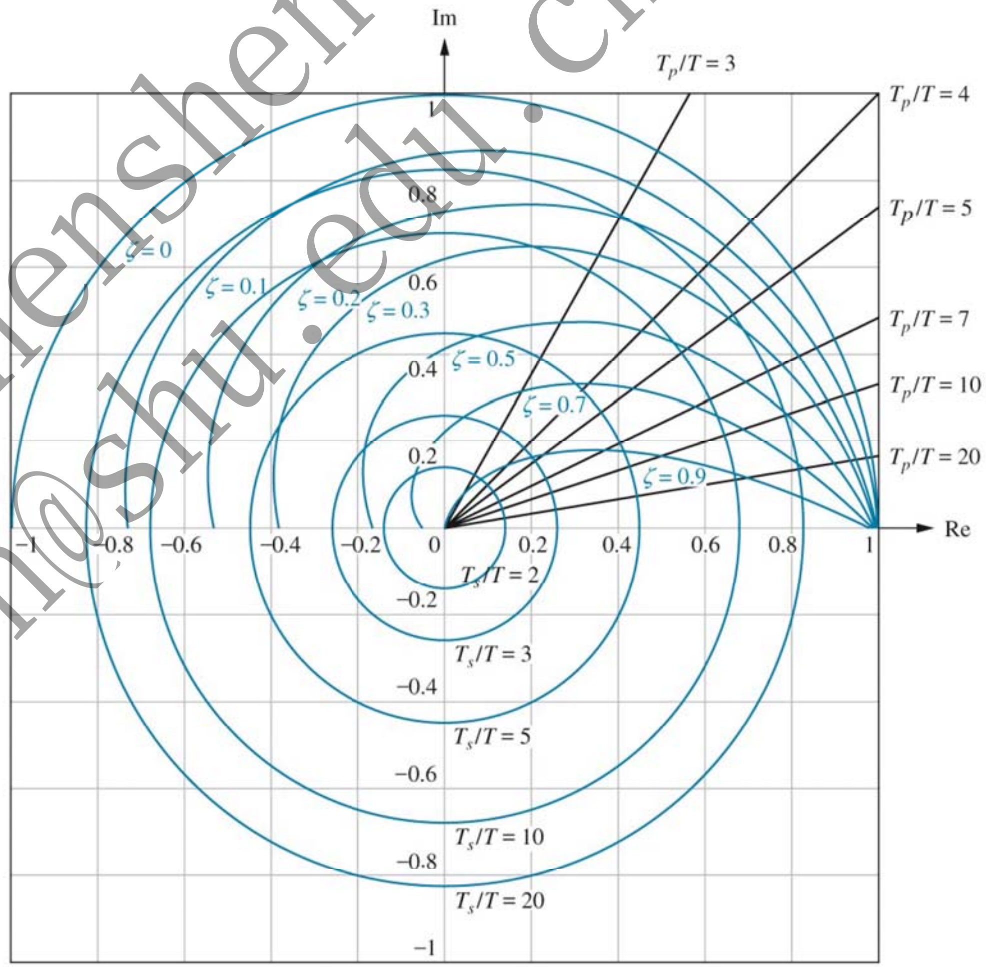


Figure 13.18

Horizontal lines

- The horizontal lines are lines of constant peak time.
- The lines are characterized by the equation $s=\sigma_1+j\omega$, where the imaginary part, $\omega_1=\pi/T_p$, is constant.
- Substituting this into $z=e^{sT}$, we obtain

$$z = e^{\sigma T} e^{j\omega_1 T} = e^{\sigma T} e^{j\theta_1}$$

- It represents radial lines at an angle of θ_1.
- If σ is negative, that section of the radial line lies inside the unit circle.
- If σ is positive, that section of the radial line lies outside the unit circle.

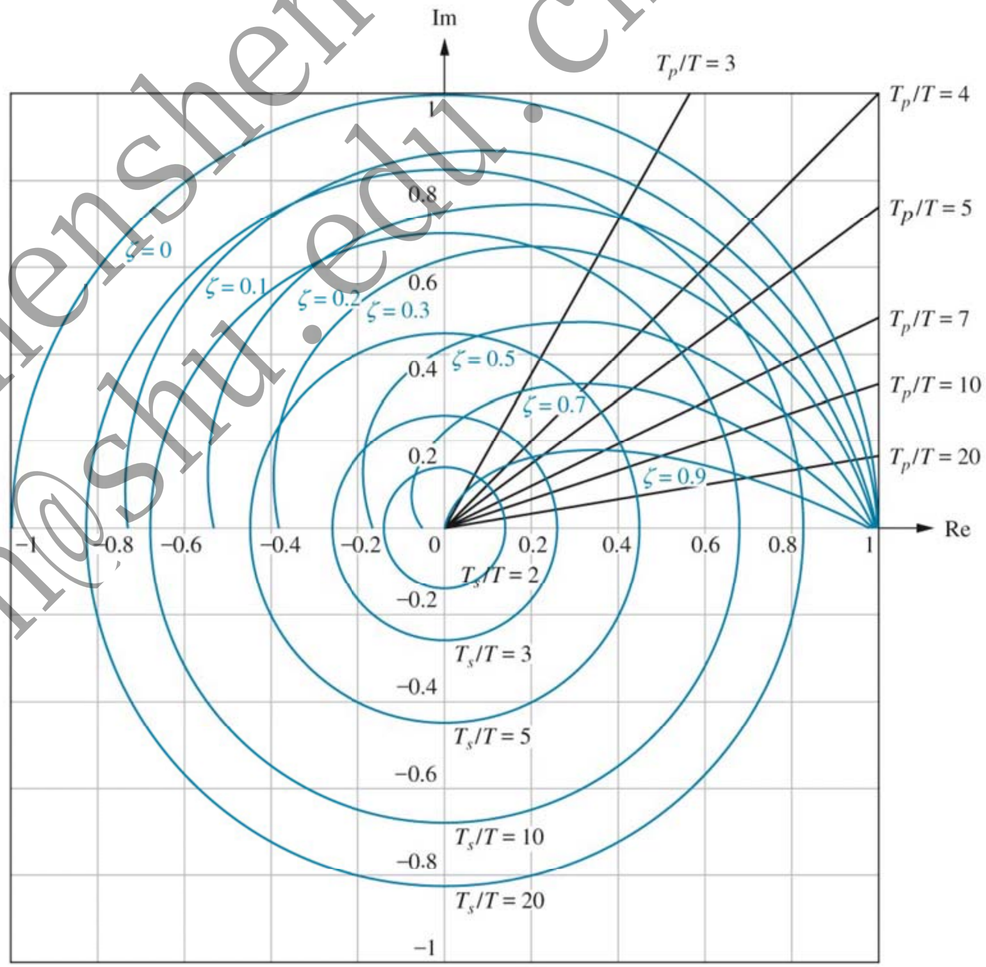


Figure 13.18

Radial lines

- The radial lines are lines of constant percent overshoot on the s-plane.
- These radial lines are represented by

$$\frac{\sigma}{\omega} = -\tan\left(\sin^{-1}\zeta\right) = -\frac{\zeta}{\sqrt{1-\zeta^2}}$$

- Hence, $s = \sigma + j\omega = -\omega\frac{\zeta}{\sqrt{1-\zeta^2}} + j\omega$

- Transforming it to the z-plane yields

$$z = e^{sT} = e^{-\omega T\left(\zeta/\sqrt{1-\zeta^2}\right)}e^{j\omega T} = e^{-\omega T\left(\zeta/\sqrt{1-\zeta^2}\right)}\angle\omega T$$

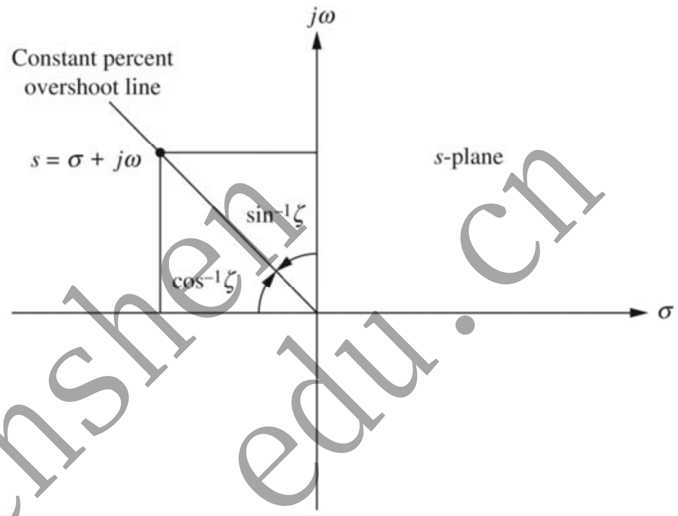


Figure 13.19

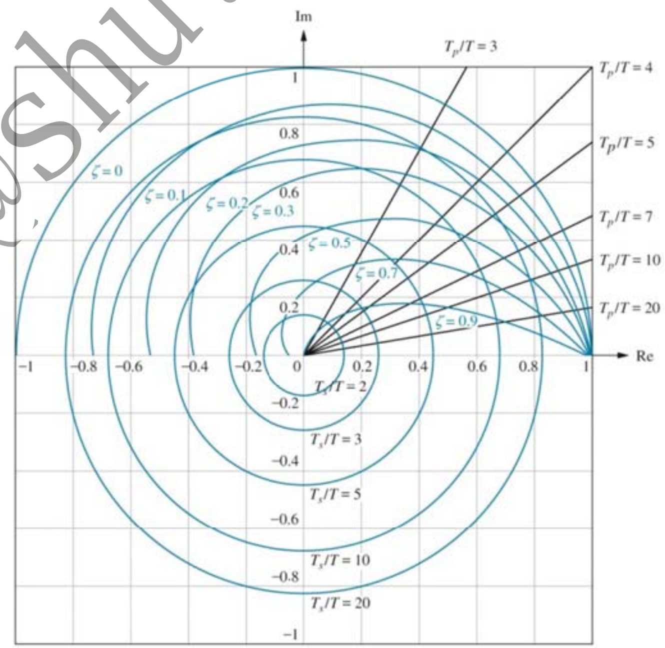


Figure 13.18

Summary

- In this chapter, we covered the design of digital systems using classical methods.
- We looked at the advantages of digital control systems. These systems can control numerous loops at reduced cost. System modifications can be implemented with software changes rather than hardware changes.
- Throughout the chapter, we saw direct parallels to the methods used for s-plane analysis of transients, steady-state errors, and the stability of analog systems.
- The parallel is made possible by the z-transform, which replaces the Laplace transform as the transform of choice for analyzing sampled-data systems. The z-transform allows us to represent sampled waveforms at the sampling instants. We can handle sampled systems as easily as continuous systems, including block diagram reduction, since both signals and systems can be represented in the z-domain and manipulated algebraically. Complex systems can be reduced to a single block through techniques that parallel those used with the s-plane. Time responses can be obtained through division of the numerator by the denominator without the partial-fraction expansion required in the s-domain.

Summary (Cont.)

- Digital systems analysis parallels the s-plane techniques in the area of stability. The unit circle becomes the boundary of stability, replacing the imaginary axis.
- We also found that the concepts of root locus and transient response are easily carried into the z-plane. The rules for sketching the root locus do not change. We can map points on the s-plane into points on the z-plane and attach transient response characteristics to the points. Evaluating a sampled-data system shows that the sampling rate, in addition to gain and load, determines the transient response.