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Principle of Automatic Control (2)
自动控制原理2 (全英语教学课程)

Topic 1
Design via Frequency Response
(Chapter 11 in text book)

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Learning Outcomes for Topic 1

After completing this topic, you will be able to:

- Use frequency response techniques to adjust the gain to meet a transient response specification (Gain adjustment);
- Use frequency response techniques to design cascade compensators to improve the steady-state error (Lag compensation);
- Use frequency response techniques to design cascade compensators to improve the transient response (Lead compensation);
- Use frequency response techniques to design cascade compensators to improve both the steady-state error and the transient response (Lag-Lead compensation).

Outline

- Brief Introduction
- Transient Response via Gain Adjustment
- Lag Compensation
- Lead Compensation
- Lag-Lead Compensation

New terminologies in this topic

- Gain margin (G_M) 增益裕度
- Phase margin (Φ_M) 相角裕度
- Lag Compensation 滞后补偿
- Lead Compensation 超前补偿
- Lag-Lead Compensation 超前-滞后补偿
- Bandwidth 带宽
- Attenuate 衰减
- Low-pass filter 低通滤波器
- High-pass filter 高通滤波器

1. Brief Introduction

- The concepts of stability, transient response, and steady-state error that we learned from last term -> designing via frequency response methods.
  - 1. Stability: The Nyquist criterion tells us how to determine if a system is stable.
  - 2. Transient response: Percent overshoot is reduced by increasing the phase margin, and the speed of the response is increased by increasing the bandwidth.
  - 3. Steady-state error: Steady-state error is improved by increasing the low-frequency magnitude responses, even if the high-frequency magnitude response is attenuated.
- The emphasis in this topic is on the design of *lag, lead, and lag-lead compensation*.

Basic Knowledge 1:Gain Margin and Phase Margin via the Nyquist Diagram

- Now that we know how to sketch and interpret a Nyquist diagram to determine a closed-loop system’s stability, let us extend our discussion to concepts that will eventually lead us to the design of transient response characteristics via frequency response techniques.
- Using the Nyquist diagram, we define two quantitative measures of how stable a system is. These quantities are called gain margin and phase margin.
- Systems with greater gain and phase margins can withstand greater changes in system parameters before becoming unstable.

- Gain margin, (G_M) . The gain margin is the change in open-loop gain, expressed in decibels (dB), required at 180 of phase shift to make the closed-loop system unstable.
- Phase margin, (Φ_M). The phase margin is the change in open-loop phase shift required at unity gain to make the closed-loop system unstable.

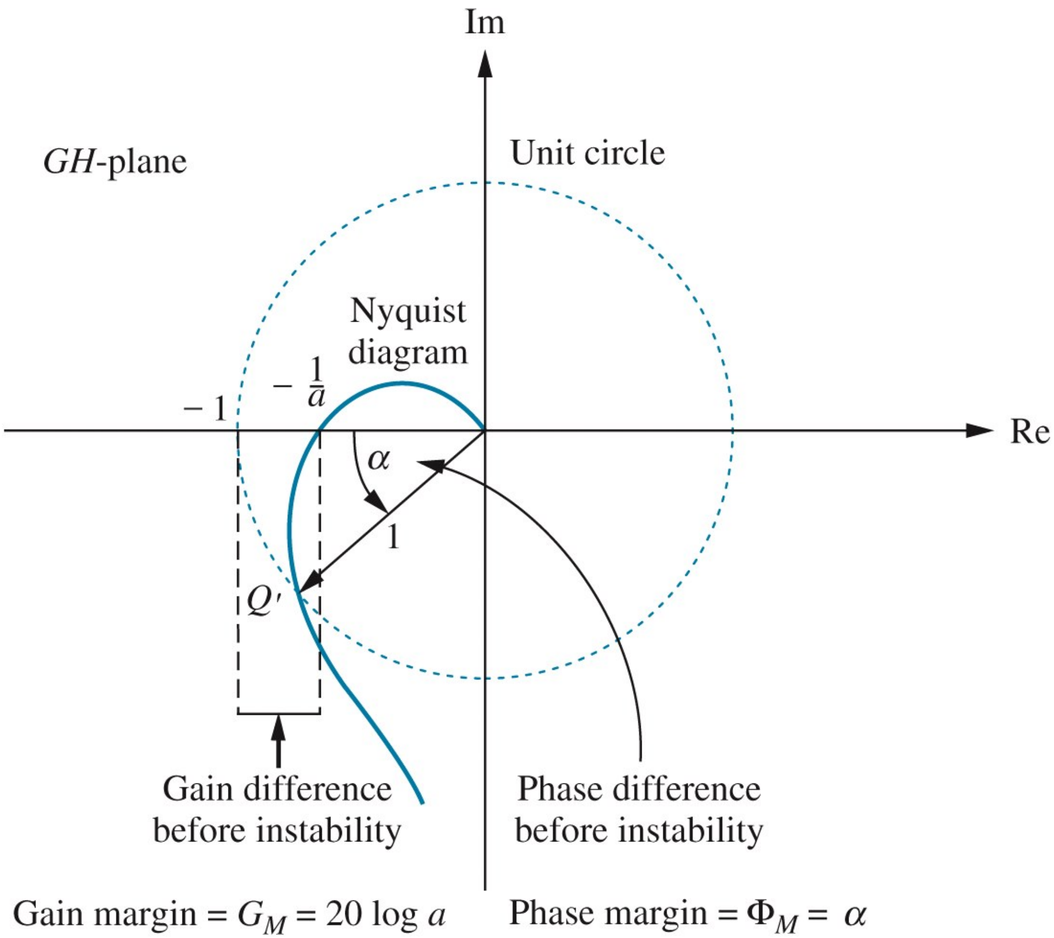


Figure 10.35

- Here a gain difference between the Nyquist diagram's crossing of the real axis at ***-1/a*** and the ***-1*** critical point determines the proximity of the system to instability. Thus, if the gain of the system were multiplied by a units, the Nyquist diagram would intersect the critical point. We then say that the ***gain margin*** is *a* units, or, expressed in dB, $G_M = 20\log a$.
- At point Q', where the gain is unity, ***α*** represents the system's proximity to instability. That is, at unity gain, if a phase shift of ***α*** degrees occurs, the system becomes unstable. Hence, the amount of ***phase margin*** is ***α***.

- Later, we show that phase margin can be related to the damping ratio. Thus, we will be able to relate frequency response characteristics to transient response characteristics as well as stability. We will also show that the calculations of gain and phase margins are more convenient if Bode plots are used rather than a Nyquist diagram

Example 10.8

Finding Gain and Phase Margins

PROBLEM: Find the gain and phase margin for the system of Example 10.7 if $K = 6$.

SOLUTION: To find the gain margin, first find the frequency where the Nyquist diagram crosses the negative real axis. Finding $G(j\omega)H(j\omega)$, we have

$$\begin{aligned} G(j\omega)H(j\omega) &= \left.\frac{6}{(s^2+2s+2)(s+2)}\right|_{s\rightarrow j\omega} \\ &= \frac{6[4(1-\omega^2)-j\omega(6-\omega^2)]}{16(1-\omega^2)^2+\omega^2(6-\omega^2)^2} \end{aligned} \quad (10.47)$$

The Nyquist diagram crosses the real axis at a frequency of $\sqrt{6}$ rad/s. The real part is calculated to be -0.3. Thus, the gain can be increased by $(1/0.3) = 3.33$ before the real part becomes -1. Hence, the gain margin is

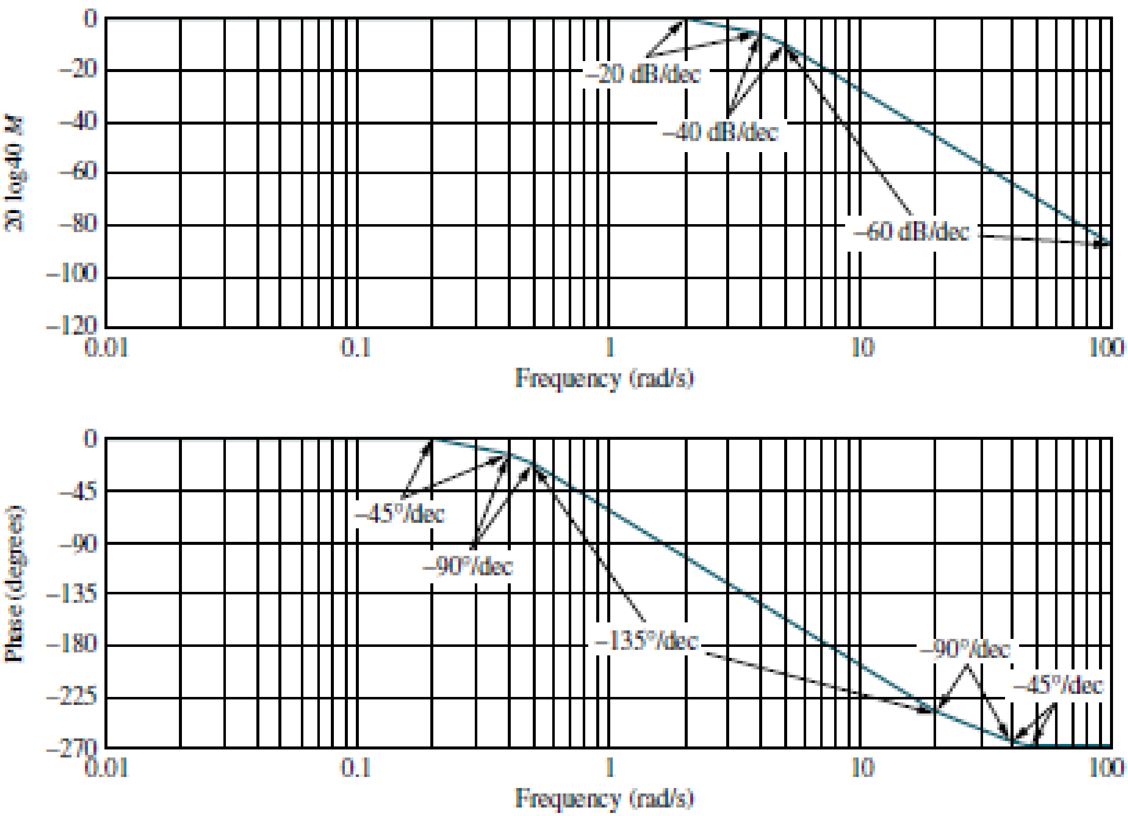
$$G_M = 20\log 3.33 = 10.45 \text{ dB} \quad (10.48)$$

To find the phase margin, find the frequency in Eq. (10.47) for which the magnitude is unity. As the problem stands, this calculation requires computational tools, such as a function solver or the program described in Appendix H.2. Later in the chapter we will simplify the process by using Bode plots. Eq. (10.47) has unity gain at a frequency of 1.253 rad/s. At this frequency, the phase angle is $-112.3°$. The difference between this angle and $-180°$ is $67.7°$, which is the phase margin.

Basic Knowledge 2:Stability, Gain Margin, and Phase Margin via Bode Plots

Determine stability via Bode Plots

G(s)=K/[(s+2)(s+4)(s+5)]



Stable -> Z=P-N=0

Since this system has all of its open-loop poles in the left–half-plane, the open-loop system is stable. Hence, the closed-loop system will be stable if the frequency response has a gain less than unity when the phase is 180°.

Evaluate Gain Margin, and Phase Margin via Bode Plots

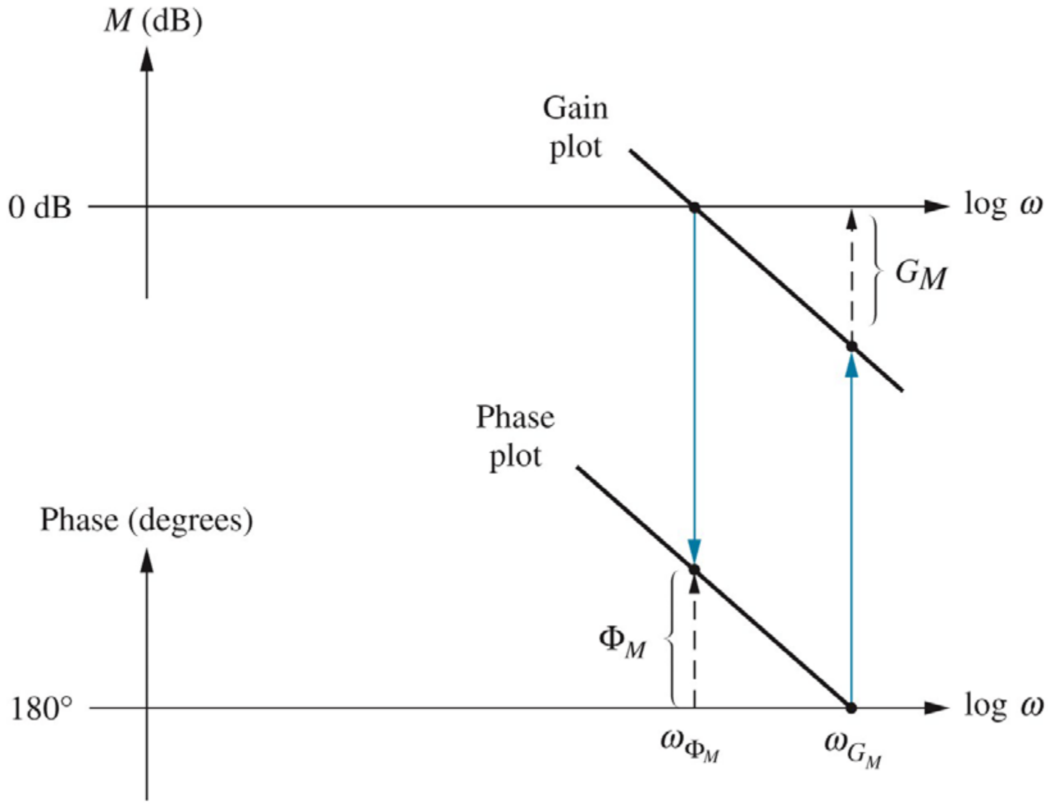


Figure 10.37

- The gain margin is found by using the phase plot to find the frequency, ω_{GM}, where the phase angle is 180. At this frequency, we look at the magnitude plot to determine the gain margin, G_M, which is the gain required to raise the magnitude curve to 0 dB.
- The phase margin is found by using the magnitude curve to find the frequency,$\omega_{\Phi M}$, where the gain is 0 dB. On the phase curve at that frequency, the phase margin, Φ_M, is the difference between the phase value and 180.

Basic Knowledge 3:Relation Between Closed-Loop Transient and Open-Loop Frequency Responses

- Damping Ratio from Phase Margin
  - Let us now derive the relationship between the phase margin and the damping ratio. This relationship will enable us to evaluate the percent overshoot from the phase margin found from the open-loop frequency response.

Consider a unity feedback system whose open-loop function

$$\frac{\omega_n^2}{s(s+2\zeta\omega_n)} \xrightarrow{\text{Closed loop transfer function}} T(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

In order to evaluate the phase margin, we let

$$|G(j\omega)|=\frac{\omega_n^2}{\left|-\omega_n^2+j2\zeta\omega_n\omega\right|}=1 \quad \Rightarrow \quad \omega_1=\omega_n\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}$$

The phase angle at this frequency is

$$\angle G(j\omega)=-90-\tan^{-1}\frac{\omega_1}{2\zeta\omega_n}=-90-\tan^{-1}\frac{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}}{2\zeta}$$

The difference between the angle and -180 is the phase margin

$$\Phi_M=90-\tan^{-1}\frac{\omega_1}{2\zeta\omega_n}=\tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2+\sqrt{1+4\zeta^4}}}$$

This figure shows the relationship between phase margin and damping ratio

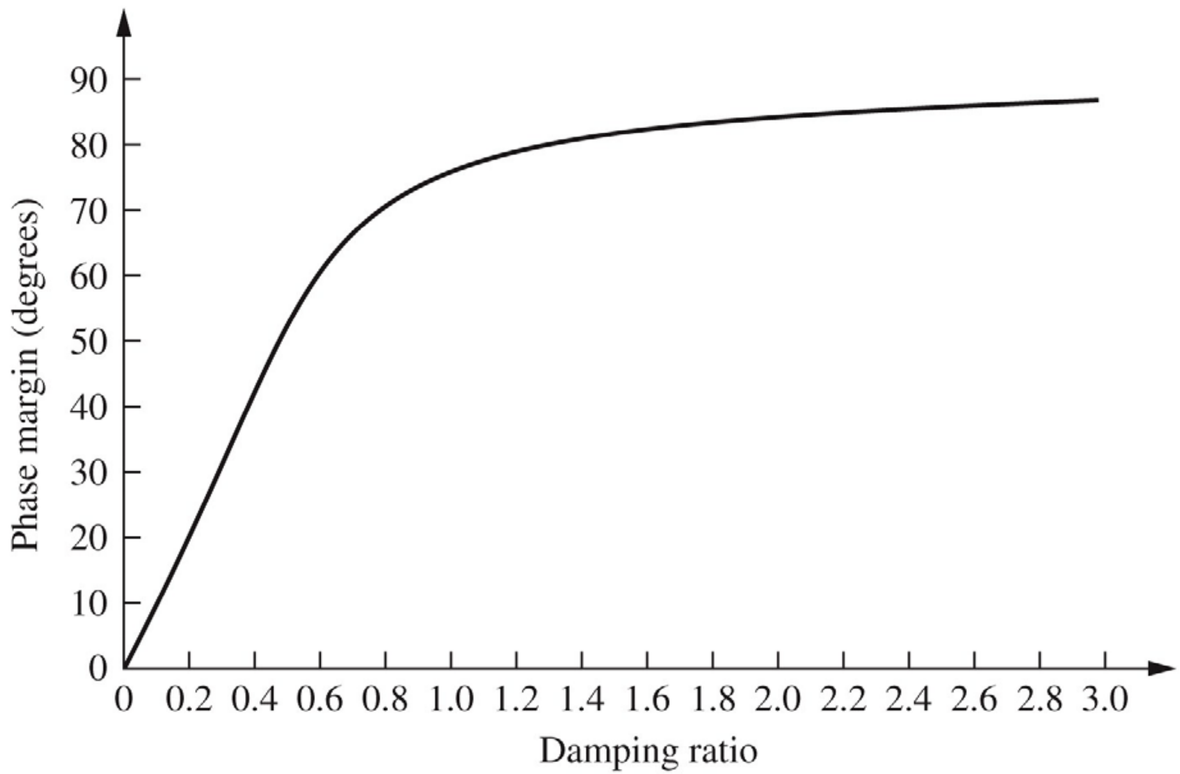
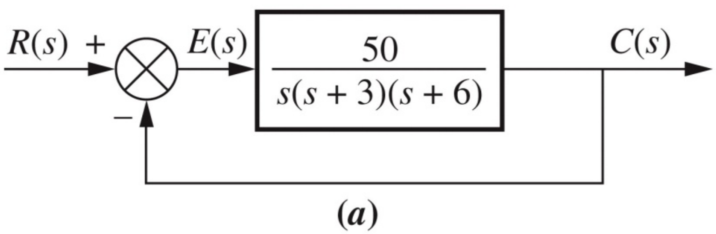


Figure 10.48

Example 10.13

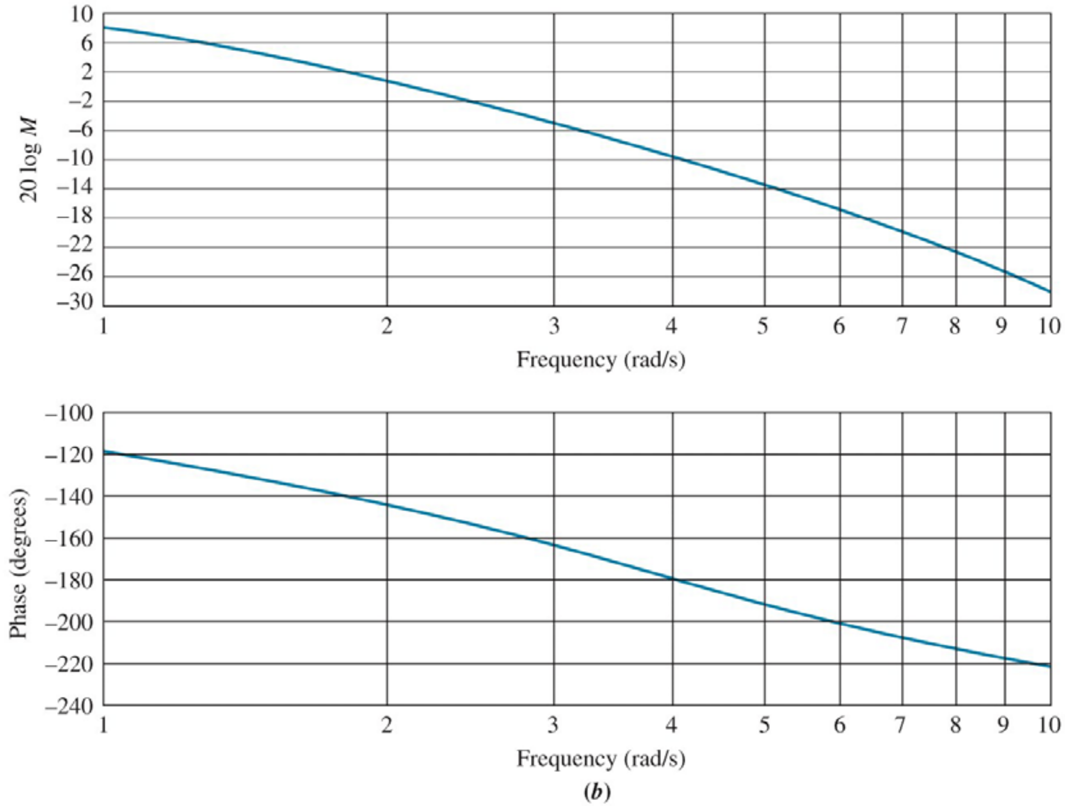
Settling and Peak Times from Open-Loop Frequency Response

PROBLEM: Given the system of Figure 10.50(*a*) and the Bode diagrams of Figure 10.50(*b*), estimate the settling time and peak time.



(*a*)

Figure 10.50a



(*b*)

Figure 10.50b

SOLUTION: Using Figure 10.50(b), we estimate the closed-loop bandwidth by finding the frequency where the open-loop magnitude response is in the range of -6 to -7.5 dB if the phase response is in the range of $-135°$ to $-225°$. Since Figure 10.50(b) shows -6 to -7.5 dB at approximately 3.7 rad/s with a phase response in the stated region, $\omega_{BW} \cong 3.7$ rad/s.

Next find ζ via the phase margin. From Figure 10.50(b), the phase margin is found by first finding the frequency at which the magnitude plot is 0 dB. At this frequency, 2.2 rad/s, the phase is about $-145°$. Hence, the phase margin is approximately $(-145° - (-180°)) = 35°$. Using Figure 10.48, $\zeta = 0.32$. Finally, using Eqs. (10.55) and (10.56), with the values of ω_{BW} and ζ just found, $T_s = 4.86$ seconds and $T_p = 129$ seconds. Checking the analysis with a computer simulation shows $T_s = 5.5$ seconds, and $T_p = 1.43$ seconds.

2. Transient Response via Gain Adjustment

- Let us begin our discussion of design via frequency response methods by discussing the link between phase margin, transient response, and gain.
- The relationship between damping ratio (equivalently percent overshoot) and phase margin was derived
- Thus, if we can vary the phase margin, we can vary the percent overshoot.

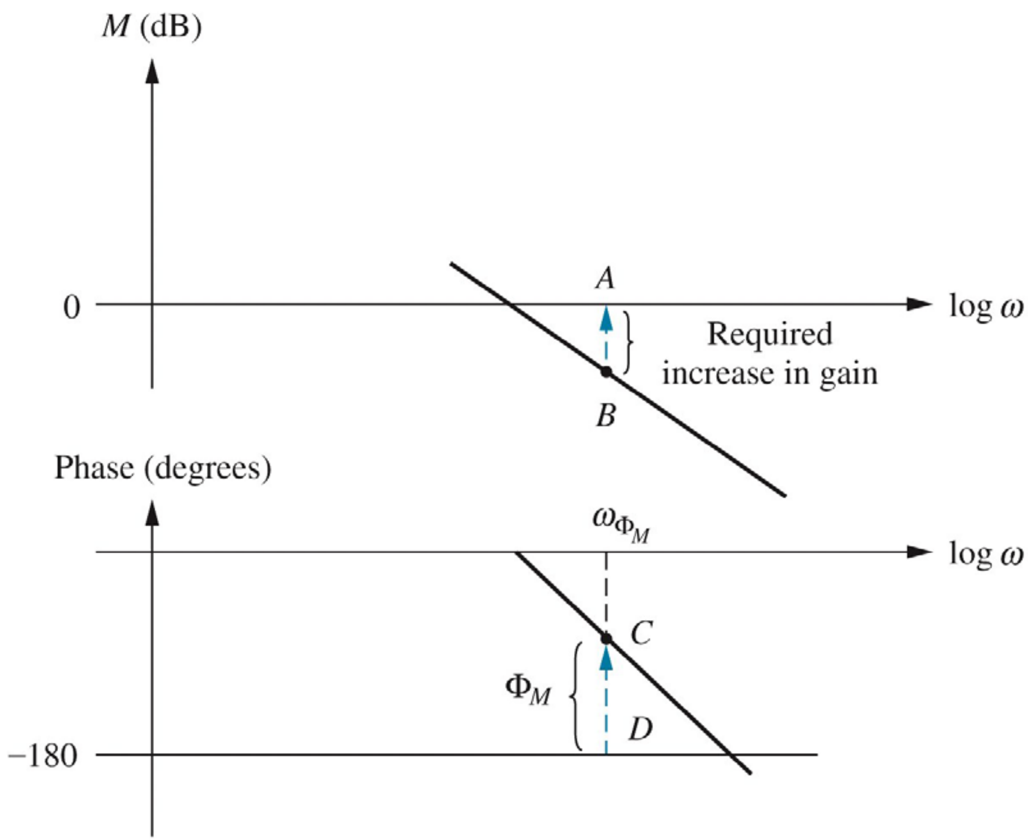


Figure 11.1

- If we desire a phase margin, Φ_M, represented by CD, we would have to raise the magnitude curve by AB. Thus, a simple gain adjustment can be used to design phase margin and, hence, percent overshoot.

Design Procedure

- 1. Draw the Bode magnitude and phase plots for a convenient value of gain.
- 2. Using the following two equations, determine the required phase margin from the percent overshoot.

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

$$\Phi_M = 90 - \tan^{-1}\frac{\omega_1}{2\zeta\omega_n} = \tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}}$$

- 3. Find the frequency, $\omega_{\Phi M}$, on the Bode phase diagram that yields the desired phase margin, CD.
- 4. Change the gain by an amount AB to force the magnitude curve to go through 0 dB at $\omega_{\Phi M}$. The amount of gain adjustment is the additional gain needed to produce the required phase margin.

Example

Transient Response Design via Gain Adjustment

PROBLEM: For the position control system shown in Figure 11.2, find the value of preamplifier gain, K, to yield a 9.5% overshoot in the transient response for a step input. Use only frequency response methods.

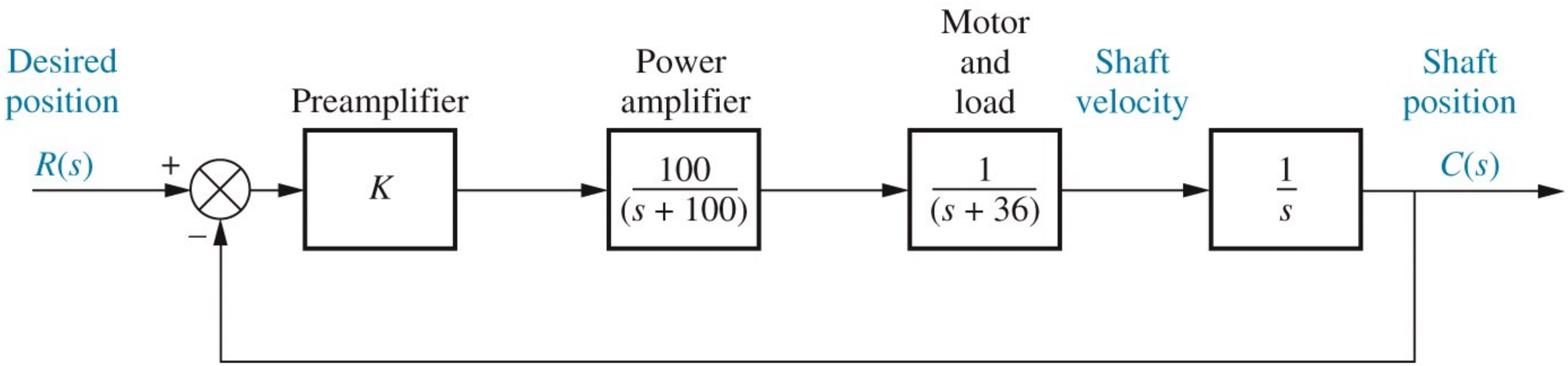


Figure 11.2

SOLUTION: We will now follow the previously described gain adjustment design procedure.

1. Choose $K = 3.6$ to start the magnitude plot at 0 dB at $\omega = 0.1$ in Figure 11.3.
2. Using Eq. (4.39), a 9.5% overshoot implies $\zeta = 0.6$ for the closed-loop dominant poles. Equation (10.73) yields a 59.2° phase margin for a damping ratio of 0.6.

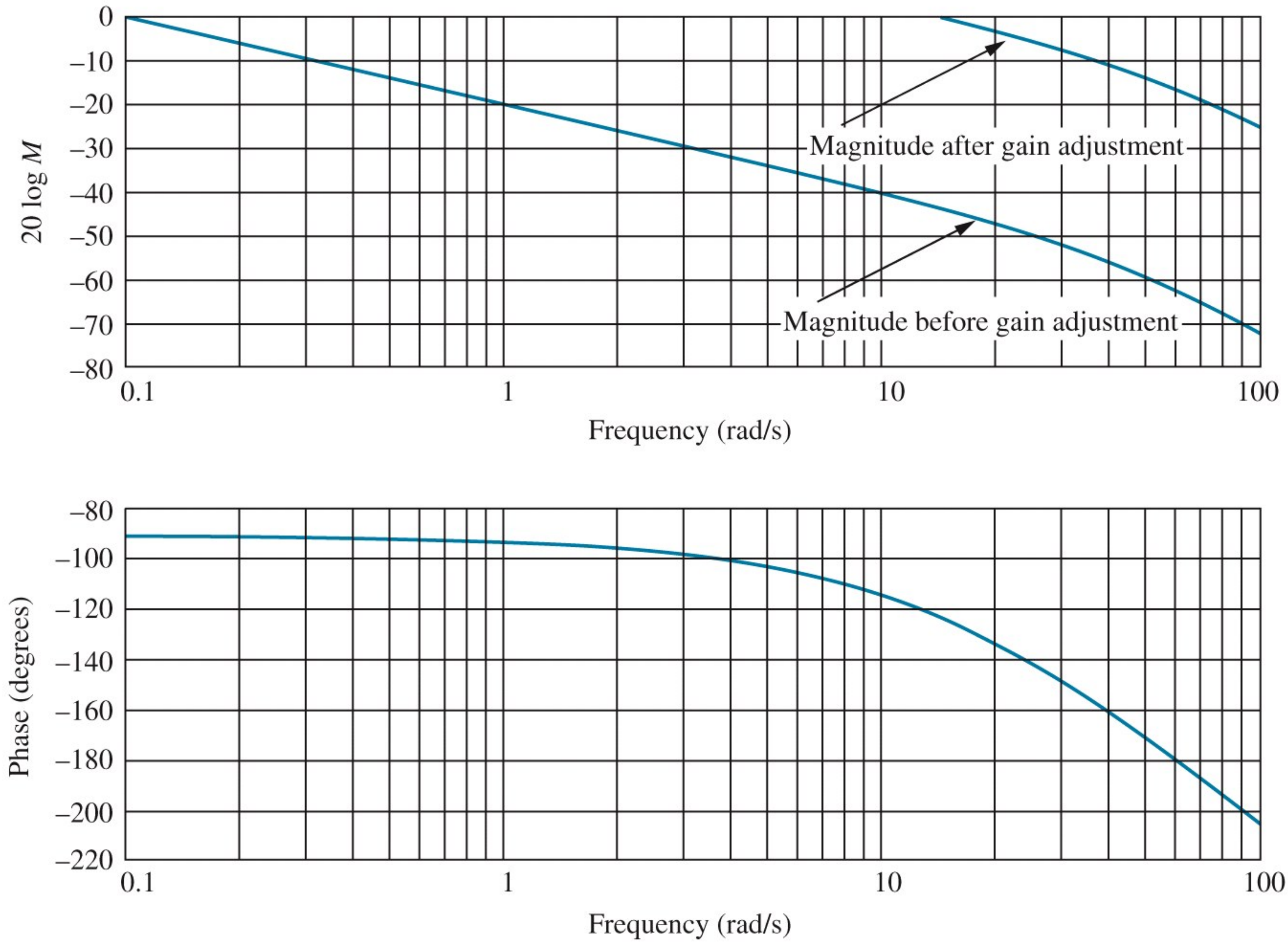


Figure 11.3

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3. Locate on the phase plot the frequency that yields a 59.2° phase margin. This frequency is found where the phase angle is the difference between −180° and 59.2°, or −120.8°. The value of the phase-margin frequency is 14.8 rad/s.

4. At a frequency of 14.8 rad/s on the magnitude plot, the gain is found to be −44.2 dB. This magnitude has to be raised to 0 dB to yield the required phase margin. Since the log-magnitude plot was drawn for $K = 3.6$, a 44.2 dB increase, or $K = 3.6 \times 162.2 = 583.9$, would yield the required phase margin for 9.48% overshoot.

The gain-adjusted open-loop transfer function is

$$G(s) = \frac{58{,}390}{s(s+36)(s+100)} \tag{11.1}$$

Table 11.1 summarizes a computer simulation of the gain-compensated system.

TABLE 11.1 Characteristic of gain-compensated system of Example 11.1

Parameter	Proposed specification	Actual value
K_v	—	16.22
Phase margin	59.2°	59.2°
Phase-margin frequency	—	14.8 rad/s
Percent overshoot	9.5	10
Peak time	—	0.18 second

Skill-Assessment Exercise

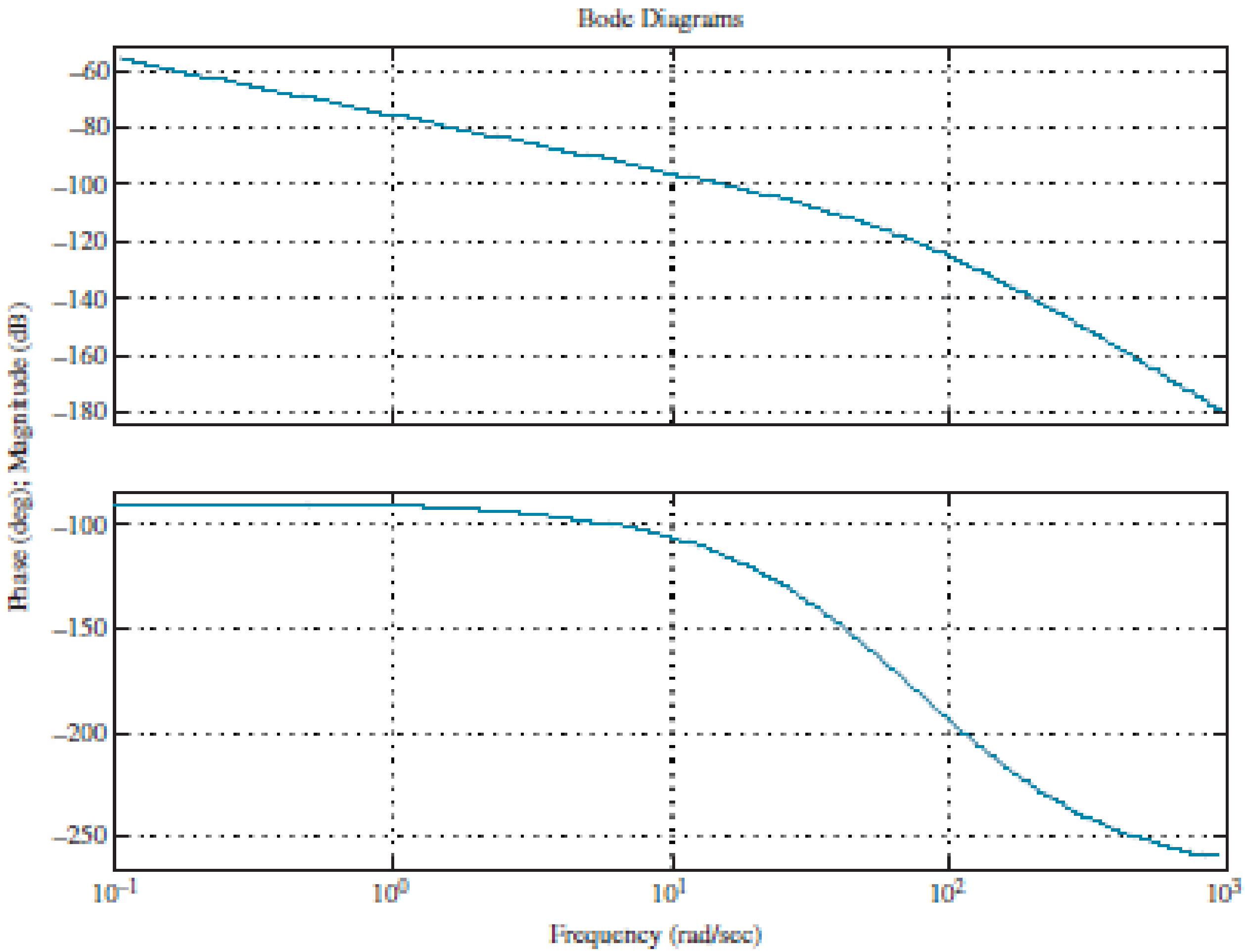
PROBLEM: For a unity feedback system with a forward transfer function

$$G(s) = \frac{K}{s(s+50)(s+120)}$$

use frequency response techniques to find the value of gain, K, to yield a closed-loop step response with 20% overshoot.

ANSWER: $K = 194{,}200$

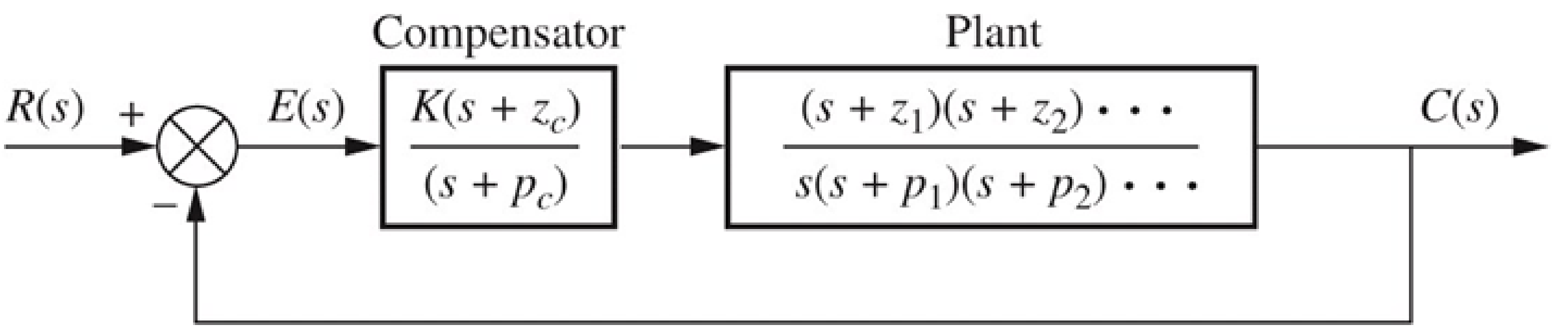
The Bode plot for $K = 1$ is shown below.



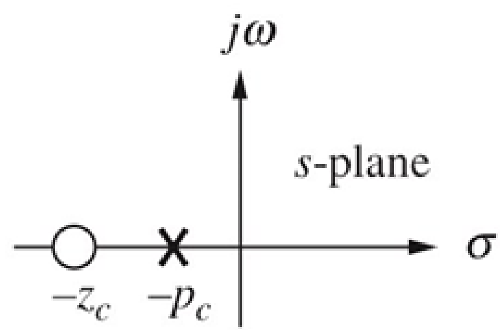
A 20% overshoot requires $\zeta = \dfrac{-\log\left(\frac{\%}{100}\right)}{\sqrt{\pi^2 + \log^2\left(\frac{\%}{100}\right)}} = 0.456$. This damping ratio implies a phase margin of 48.10, which is obtained when the phase angle $= -180 \;\; + 48.10 = 131.9°$. This phase angle occurs at $\omega = 27.6$ rad/s. The magnitude at this frequency is 5.15×10^{-6}. Since the magnitude must be unity $K = \dfrac{1}{5.15 \times 10^{-6}} = 194,200$.

3. Lag Compensation

- Lag compensator: A transfer function, characterized by a pole on the negative real axis close to the origin and a zero close and to the left of the pole, that is used for the purpose of improving the steady-state error of a closed-loop system.
- The function of the lag compensator as seen on Bode diagrams is to
  - (1) improve the static error constant by increasing only the low-frequency gain without any resulting instability, and
  - (2) increase the phase margin of the system to yield the desired transient response.



$$G_c(s) = \frac{(s + z_c)}{(s + p_c)}$$



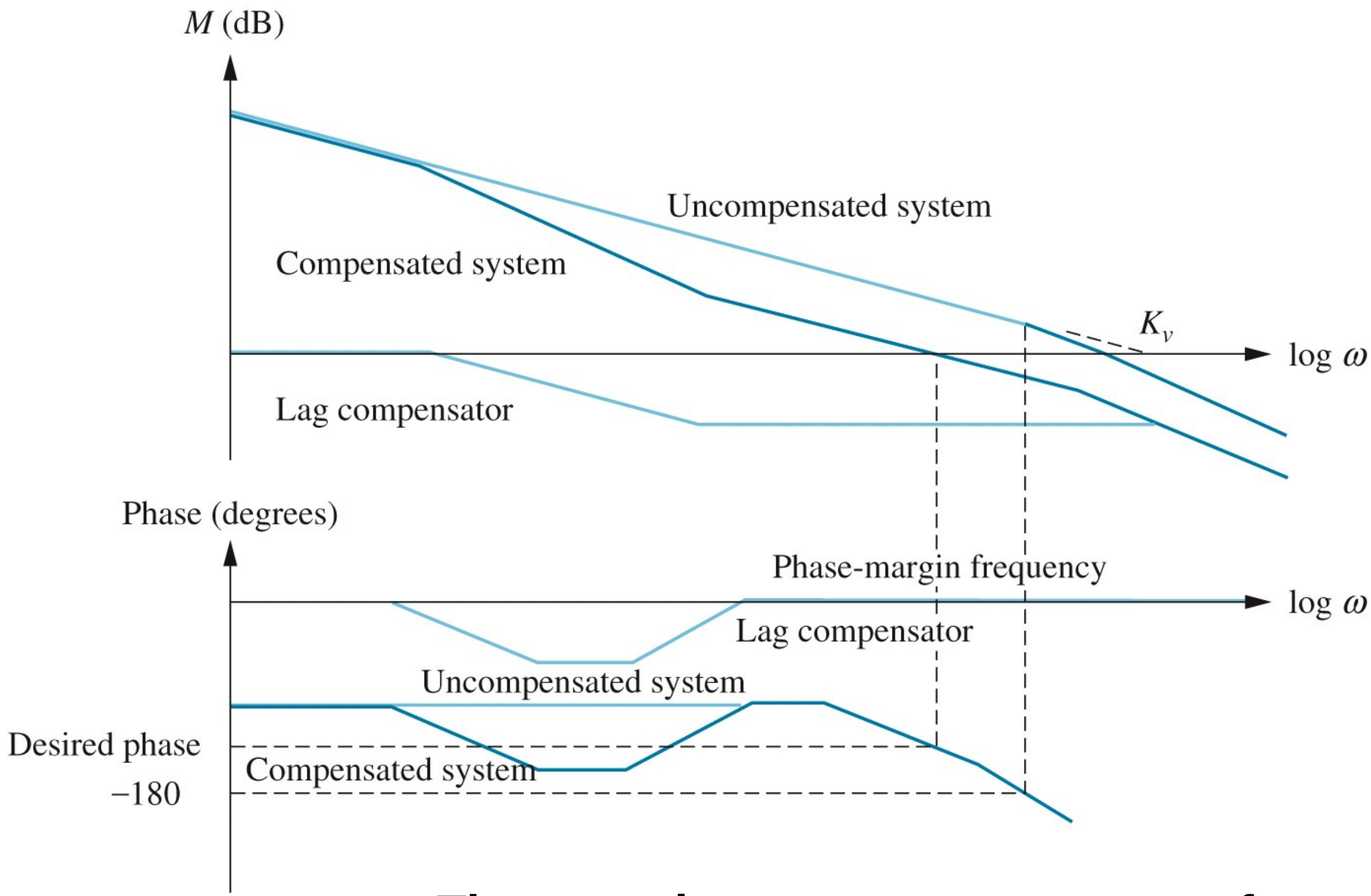


Figure 11.4

The name lag compensator comes from the fact that the typical phase angle response for the compensator, as shown in Figure, is always negative, or lagging in phase angle.

For detail please refer to textbook from page 593 to page 595

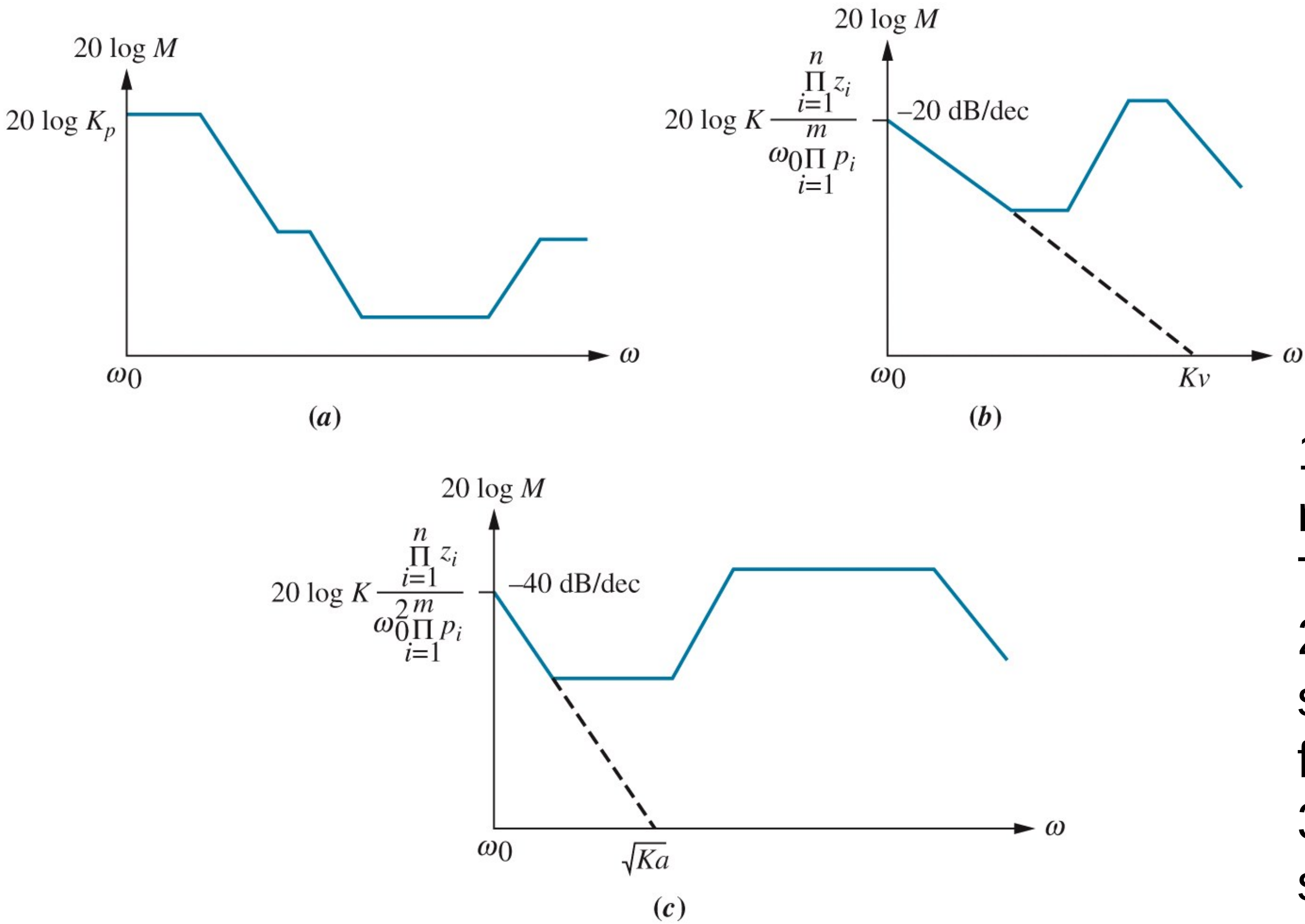


Figure 10.51

1. The low-frequency magnitude is *20logKp* for a Type 0 system;
2. The initial -20dB/decade slope intersects the frequency axis at *Kv;*
3. The initial -40dB/decade slope intersects the frequency axis at √*Ka.*

Design Procedure (Four Steps)

- 1. Set the gain, ***K***, to the value that satisfies the steady-state error specification and plot the Bode magnitude and phase diagrams for this value of gain.
- 2. Find the frequency where the phase margin is ***5°*** to ***12°*** greater than the phase margin that yields the desired transient response. This step compensates for the fact that the phase of the lag compensator may still contribute anywhere from ***-5°*** to ***-12°*** of phase at the phase-margin frequency.
- This value of break frequency ensures that there will be only -5 to -12 phase contribution from the compensator at the frequency found in Step 2.

- 3. Select a lag compensator whose magnitude response yields a composite Bode magnitude diagram that goes through ***0dB*** at the frequency found in Step 2 as follows: Draw the compensator's high-frequency asymptote to yield ***0dB*** for the compensated system at the frequency found in Step 2. Thus, if the gain at the frequency found in Step 2 is ***20logK$_{PM}$***, then the compensator's high-frequency asymptote will be set at ***-20logK$_{PM}$***; select the upper break frequency to be ***1*** decade below the frequency found in Step 2; select the low-frequency asymptote to be at ***0dB***; connect the compensator's high- and low-frequency asymptotes with a ***-20dB/decade*** line to locate the lower break frequency.
- 4. Reset the system gain, ***K***, to compensate for any attenuation in the lag network in order to keep the static error constant the same as that found in Step 1.

- We are relying upon the initial gain setting to meet the steady-state requirements and then relying upon the lag compensator’s -20 dB/decade slope to meet the transient response requirement by setting the 0 dB crossing of the magnitude plot.
- The transfer function of the lag compensator is

$$G_c(s) = \frac{s + \frac{1}{T}}{s + \frac{1}{\alpha T}} \qquad (\alpha > 1)$$

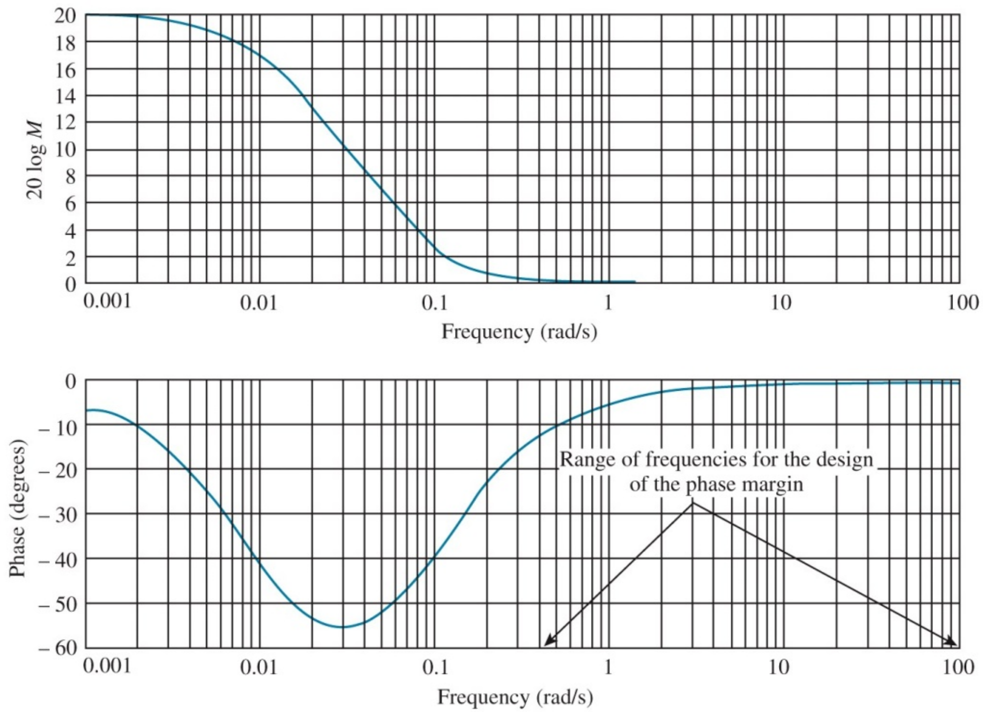


Figure 11.5

- Figure shows the frequency response curves for the lag compensator. The range of high frequencies shown in the phase plot is where we will design our phase margin. This region is after the second break frequency of the lag compensator, where we can rely on the attenuation characteristics of the lag network to reduce the total open-loop gain to unity at the phase-margin frequency.

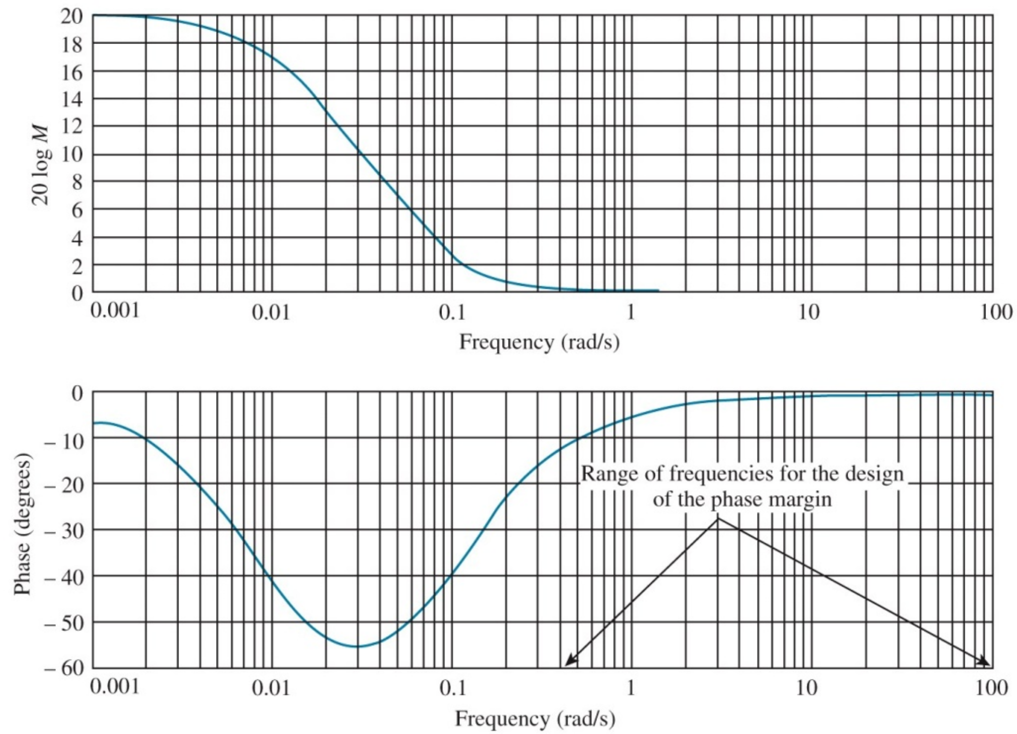


Figure 11.5

- Further, in this region the phase response of the compensator will have minimal effect on our design of the phase margin. Since there is still some effect, approximately 5 ° to 12 ° , we will add this amount to our phase margin to compensate for the phase response of the lag compensator (see Step 2).

Lag Compensation Design

PROBLEM: Given the system of Figure 11.2, use Bode diagrams to design a lag compensator to yield a tenfold improvement in steady-state error over the gain-compensated system while keeping the percent overshoot at 9.5%.

SOLUTION: We will follow the previously described lag compensation design procedure.

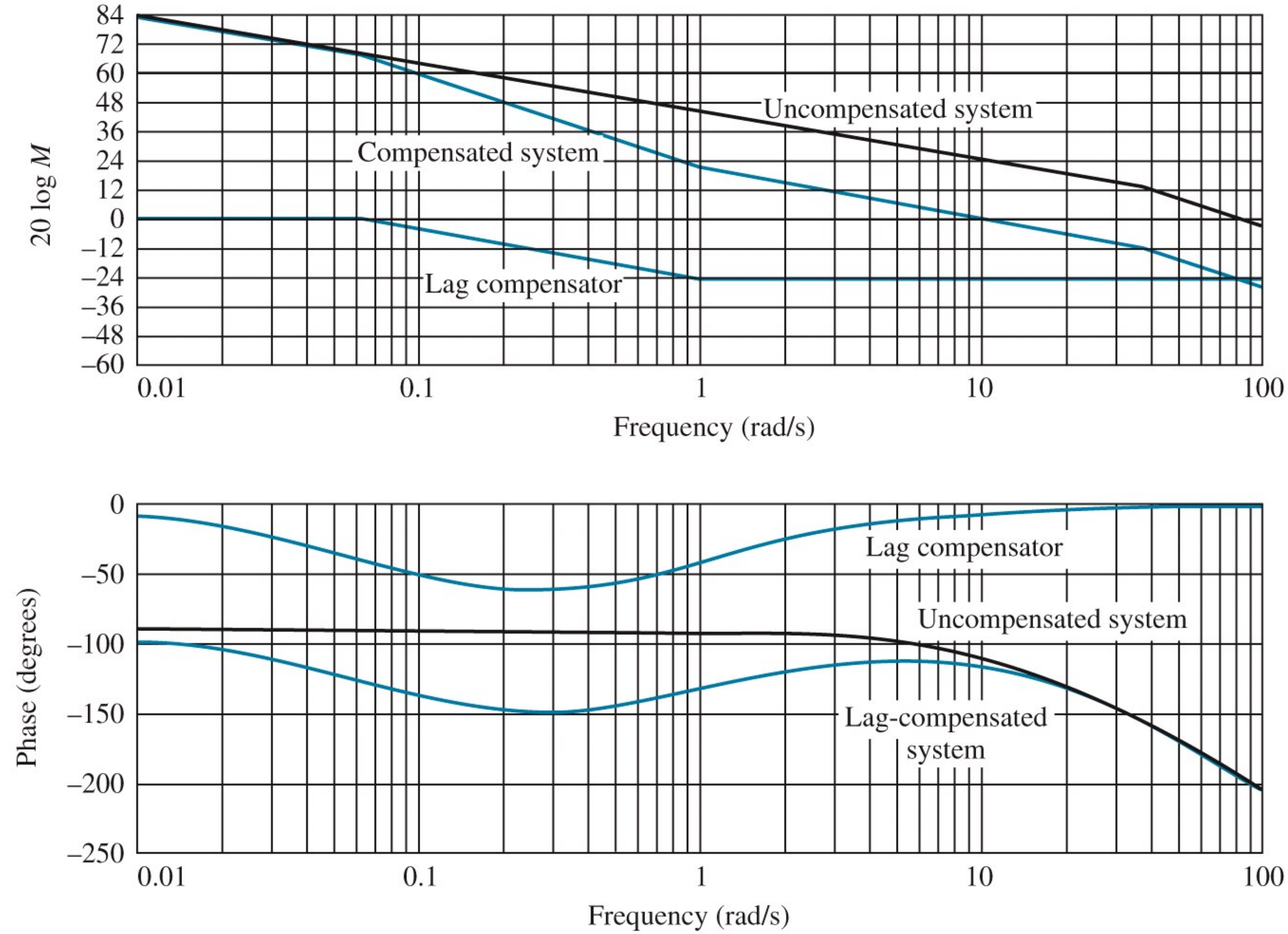


Figure 11.6

1. Set the gain, ***K***, to the value that satisfies the steady-state error specification and plot the Bode magnitude and phase diagrams for this value of gain.

1. From Example 11.1 a gain, K, of 583.9 yields a 9.5% overshoot. Thus, for this system, $K_v = 16.22$. For a tenfold improvement in steady-state error, K_v must increase by a factor of 10, or $K_v = 162.2$. Therefore, the value of K in Figure 11.2 equals 5839, and the open-loop transfer function is

$$G(s) = \frac{583{,}900}{s(s+36)(s+100)} \tag{11.3}$$

The Bode plots for $K = 5839$ are shown in Figure 11.6.

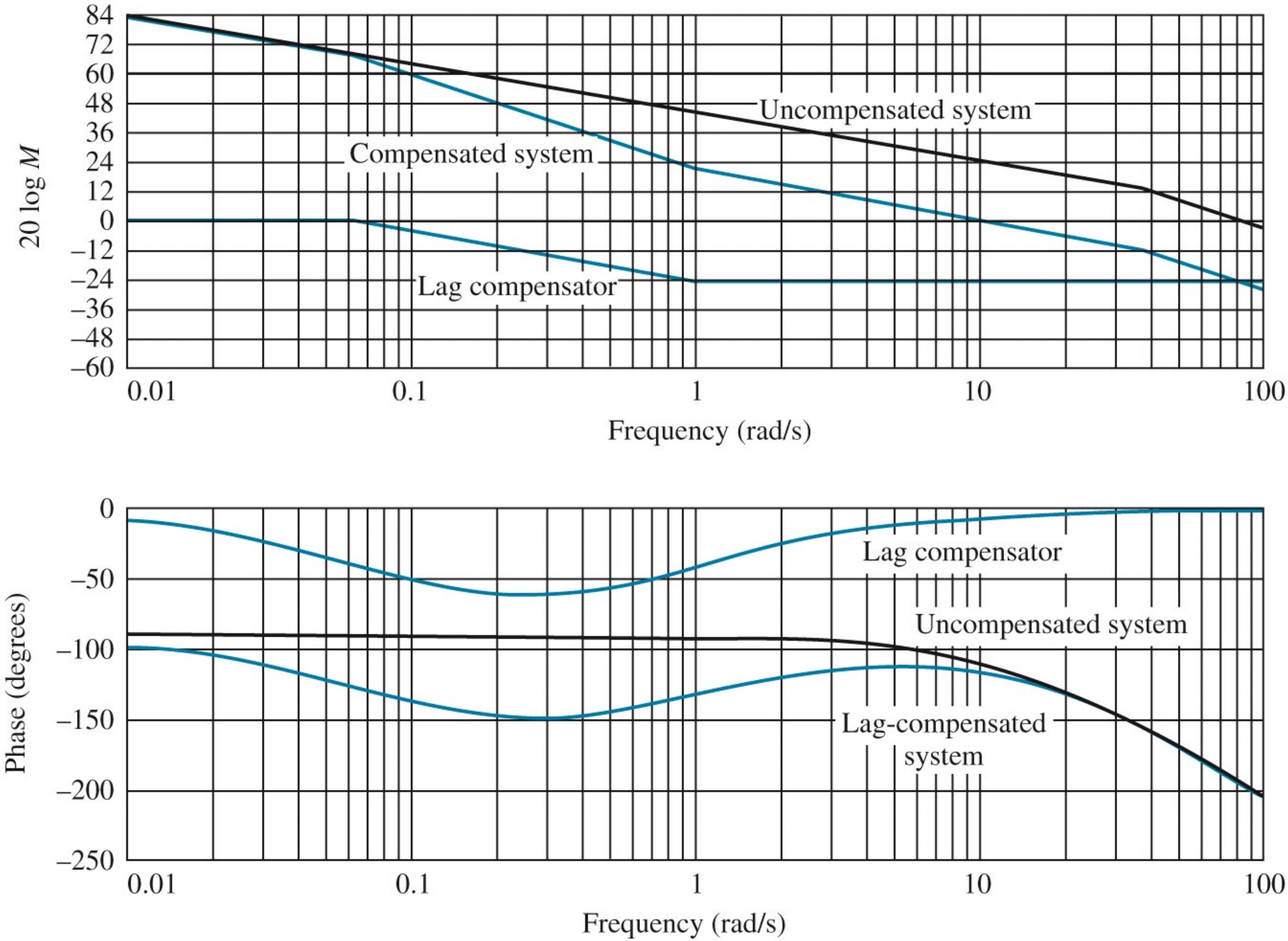

84
72
60
48
36
24
12
0
–12
–24
–36
–48
–60
20 log M
Uncompensated system
Compensated system
Lag compensator
0.01
0.1
1
10
100
Frequency (rad/s)
0
–50
–100
–150
–200
–250
Phase (degrees)
Lag compensator
Uncompensated system
Lag-compensated system
0.01
0.1
1
10
100
Frequency (rad/s)

Figure 11.6

2. Find the frequency where the phase margin is ***5°*** to ***12°*** greater than the phase margin that yields the desired transient response. This step compensates for the fact that the phase of the lag compensator may still contribute anywhere from ***-5°*** to ***-12°*** of phase at the phase-margin frequency.

2. The phase margin required for a 9.5% overshoot ($\zeta = 0.6$) is found from Eq. (10.73) to be 59.2°. We increase this value of phase margin by 10° to 69.2° in order to compensate for the phase angle contribution of the lag compensator. Now find the frequency where the phase margin is 69.2°. This frequency occurs at a phase angle of $-180° + 69.2° = -110.8°$ and is 9.8 rad/s. At this frequency, the magnitude plot must go through 0 dB. The magnitude at 9.8 rad/s is now +24 dB (exact, that is, nonasymptotic). Thus, the lag compensator must provide −24 dB attenuation at 9.8 rad/s.

3. Select a lag compensator whose magnitude response yields a composite Bode magnitude diagram that goes through ***0dB*** at the frequency found in Step 2 as follows: Draw the compensator's high-frequency asymptote to yield ***0dB*** for the compensated system at the frequency found in Step 2. Thus, if the gain at the frequency found in Step 2 is $20\log K_{PM}$, then the compensator's high-frequency asymptote will be set at $-20\log K_{PM}$; select the upper break frequency to be ***1*** decade below the frequency found in Step 2; select the low-frequency asymptote to be at ***0dB***; connect the compensator's high- and low-frequency asymptotes with a ***-20dB/decade*** line to locate the lower break frequency.

4. Reset the system gain, ***K***, to compensate for any attenuation in the lag network in order to keep the static error constant the same as that found in Step 1.

3.&4. We now design the compensator. First draw the high-frequency asymptote at −24 dB. Arbitrarily select the higher break frequency to be about one decade below the phase-margin frequency, or 0.98 rad/s. Starting at the intersection of this frequency with the lag compensator's high-frequency asymptote, draw a −20 dB/decade line until 0 dB is reached. The compensator must have a dc gain of unity to retain the value of K_v that we have already designed by setting $K = 5839$. The lower break frequency is found to be 0.062 rad/s. Hence, the lag compensator's transfer function is

$$G_c(s) = \frac{0.063(s + 0.98)}{(s + 0.062)} \tag{11.4}$$

where the gain of the compensator is 0.063 to yield a dc gain of unity.

The compensated system's forward transfer function is thus

$$G(s)G_c(s) = \frac{36{,}786(s+0.98)}{s(s+36)(s+100)(s+0.062)} \quad (11.5)$$

The characteristics of the compensated system, found from a simulation and exact frequency response plots, are summarized in Table 11.2.

TABLE 11.2 Characteristics of the lag-compensated system of Example 11.2

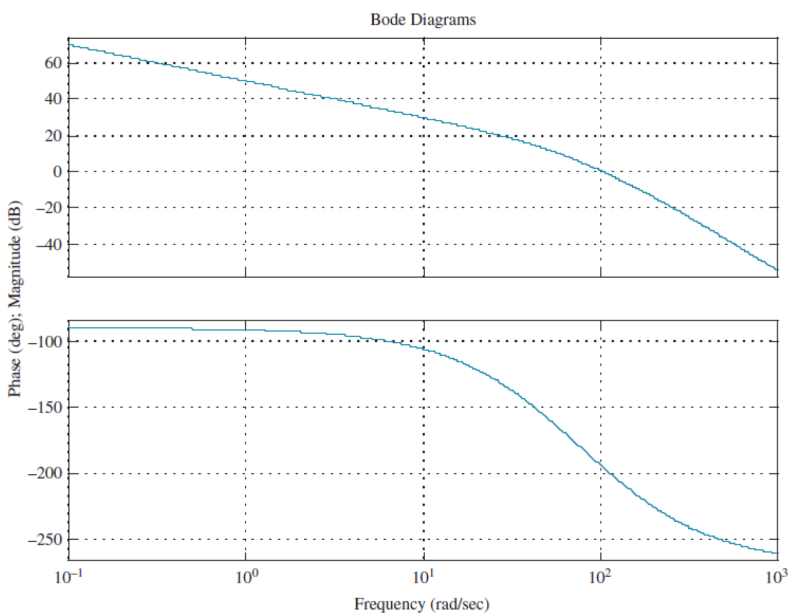
Parameter	Proposed specification	Actual value
K_v	162.2	161.5
Phase margin	59.2°	62°
Phase-margin frequency	—	11 rad/s
Percent overshoot	9.5	10
Peak time	—	0.25 second

Skill-Assessment Exercise 11.2

PROBLEM: Design a lag compensator for the system in Skill-Assessment Exercise 11.1 that will improve the steady-state error tenfold, while still operating with 20% overshoot.

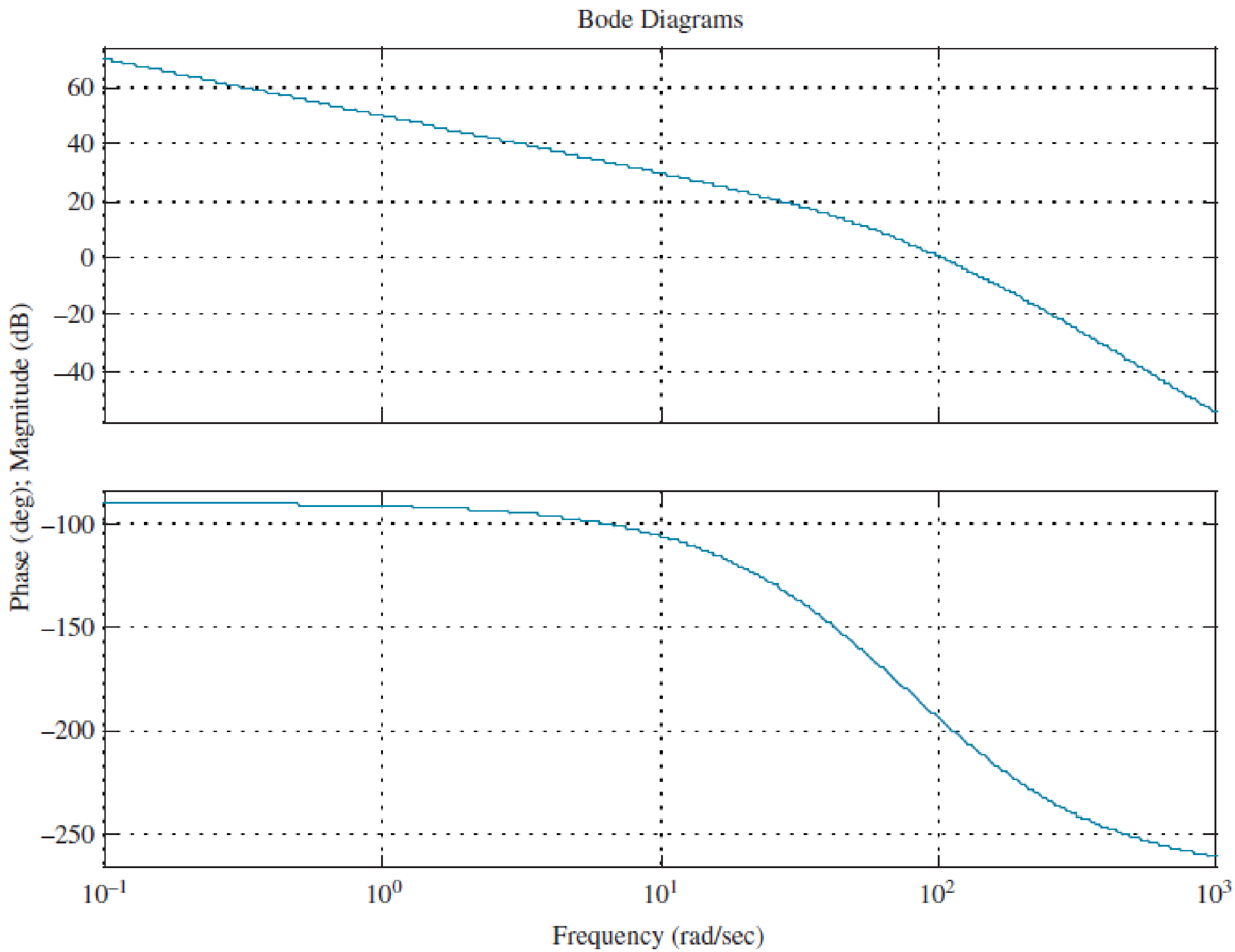
ANSWER:

$$G_{\text{lag}}(s) = \frac{0.0691(s+2.04)}{(s+0.141)}; \quad G(s) = \frac{1{,}942{,}000}{s(s+50)(s+120)}$$



K=1,942,000

To meet the steady-state error requirement, $K = 1,942,000$. The Bode plot for this gain is shown below.



A 20% overshoot requires $\zeta = \dfrac{-\log\left(\dfrac{\%}{100}\right)}{\sqrt{\pi^2 + \log^2\left(\dfrac{\%}{100}\right)}} = 0.456$. This damping ratio implies a phase margin of 48.1°. Adding 10° to compensate for the phase angle contribution of the lag, we use 58.1°. Thus, we look for a phase angle of $-180° + 58.1° = -129.9°$. The frequency at which this phase occurs is 20.4 rad/s. At this frequency the magnitude plot must go through zero dB. Presently, the magnitude plot is 23.2 dB. Therefore draw the high frequency asymptote of the lag compensator at –23.2 dB. Insert a break at $0.1(20.4) = 2.04$ rad/s. At this frequency, draw –23.2 dB/dec slope until it intersects 0 dB. The frequency of intersection will be the low frequency break or 0.141 rad/s. Hence the compensator is $G_c(s) = K_c \dfrac{(s+2.04)}{(s+0.141)}$, where the gain is chosen to yield 0 dB at low frequencies, or $K_c = 0.141/2.04 = 0.0691$. In summary,

$$G_c(s) = 0.0691\,\frac{(s+2.04)}{(s+0.141)} \text{ and } G(s) = \frac{1{,}942{,}000}{s(s+50)(s+120)}$$

4. Lead Compensation

- For second-order systems, we derived these following relationships:

phase margin & percent overshoot

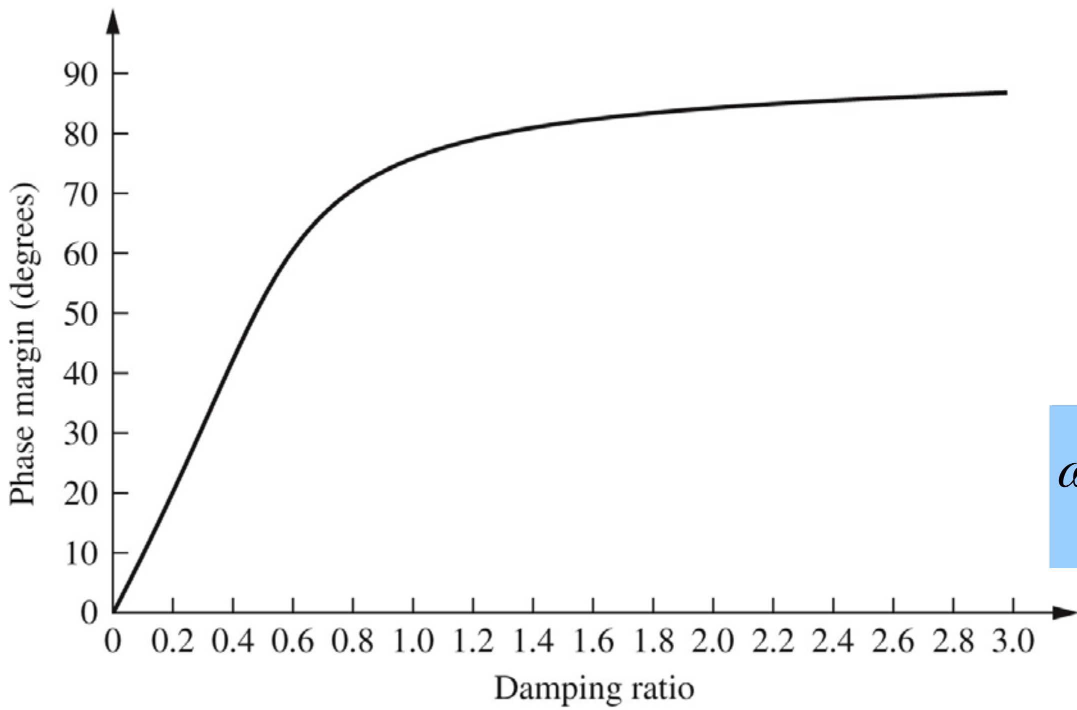


Figure 10.48

closed-loop bandwidth & other time-domain specifications

settling time:

$$\omega_{BW} = \frac{4}{T_s \zeta}\sqrt{\left(1-2\zeta^2\right)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

peak time:

$$\omega_{BW} = \frac{4}{T_p\sqrt{1-\zeta^2}}\sqrt{\left(1-2\zeta^2\right)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

- When we designed the lag network to improve the steady-state error, we wanted a minimal effect on the phase diagram in order to yield an imperceptible change in the transient response.

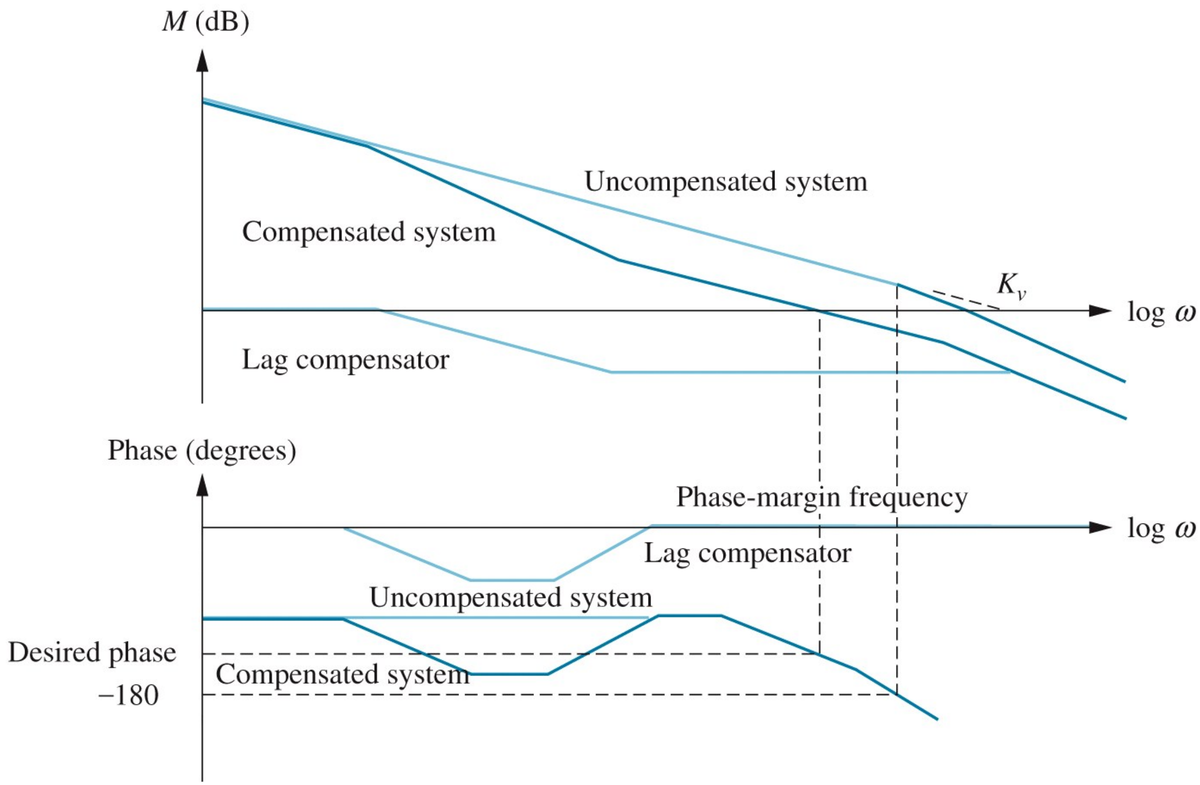


Figure 11.4

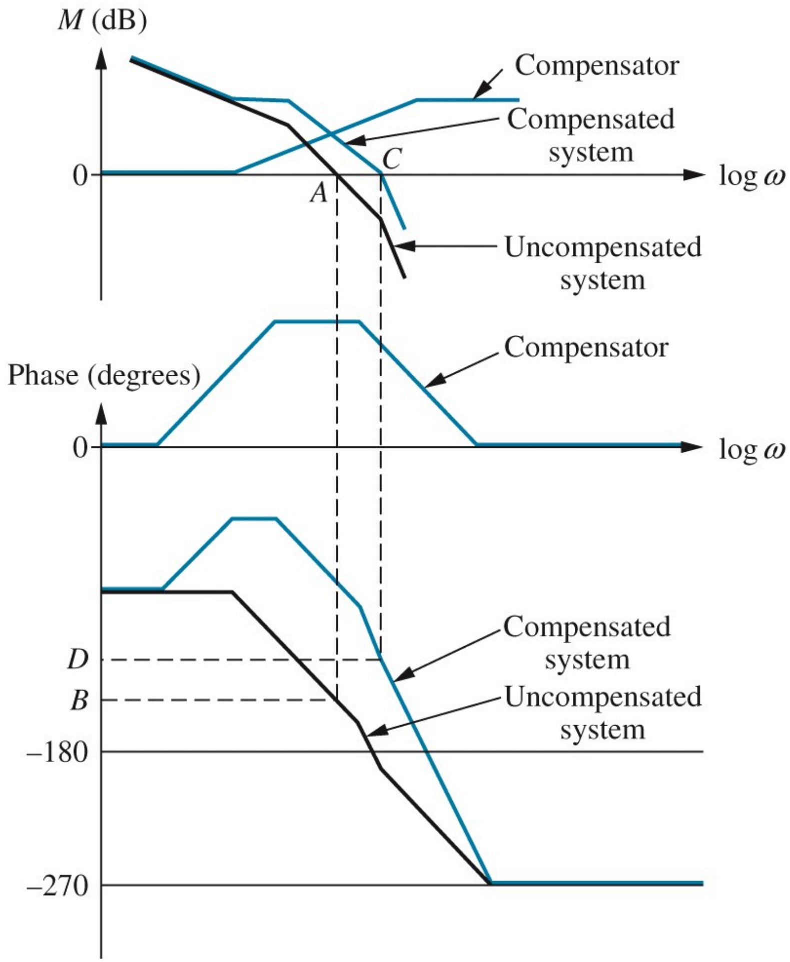
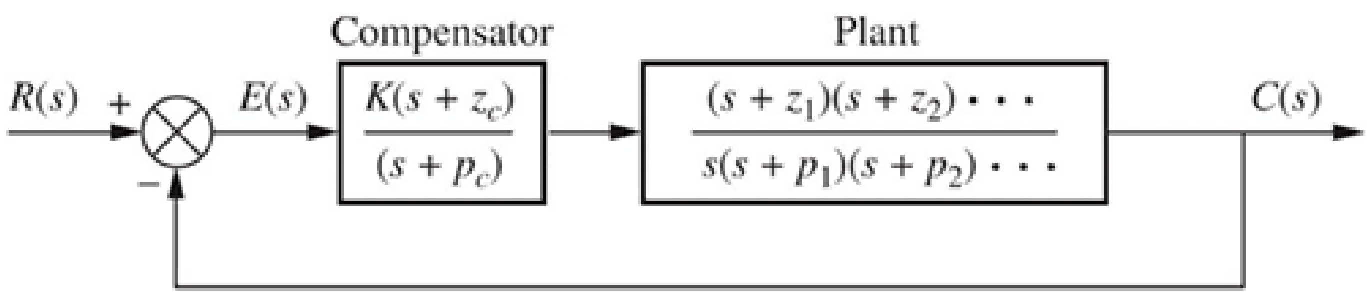


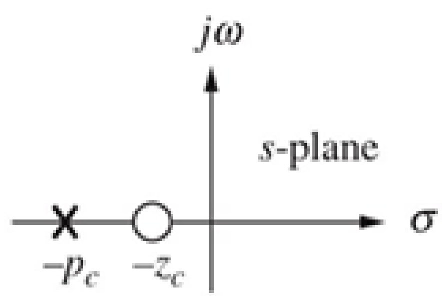
Figure 11.7

- However, in designing lead compensators via Bode plots, we want to change the phase diagram, **increasing the phase margin to reduce the percent overshoot**, and **increasing the gain crossover to realize a faster transient response**.

- Lead compensator: A transfer function, characterized by a zero on the negative real axis and a pole to the left of the zero, that is used for the purpose of improving the transient response of a closed-loop system.



$$G_c(s) = \frac{(s + z_c)}{(s + p_c)}$$



Visualizing Lead Compensation

- The lead compensator **increases the bandwidth** by increasing the gain crossover frequency.
- At the same time, the **phase diagram is raised at higher frequencies**.
- The result is a **larger phase margin** and a **higher phase-margin frequency**. In the time domain, **lower percent overshoots** (larger phase margins) with **smaller peak times** (higher phase margin frequencies/ large bandwidth) are the results.

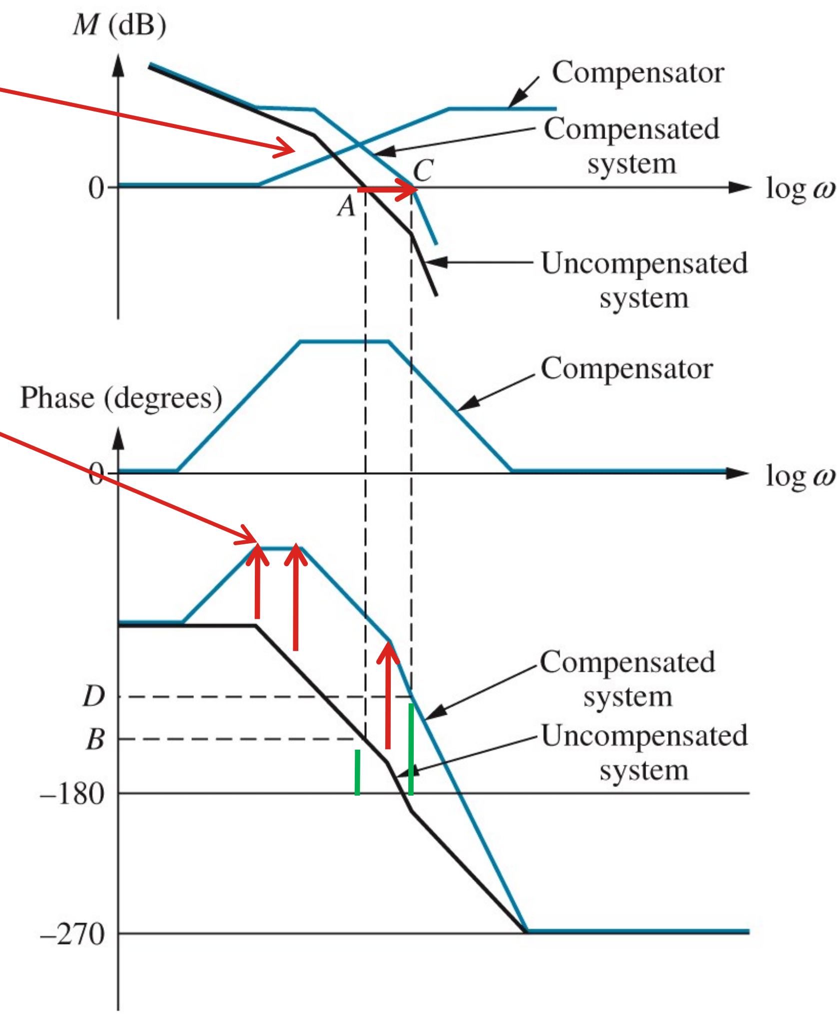


Figure 11.7

- The transfer function of the lead compensator is

$$G_c(s)=\frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} \qquad (0<\beta<1)$$

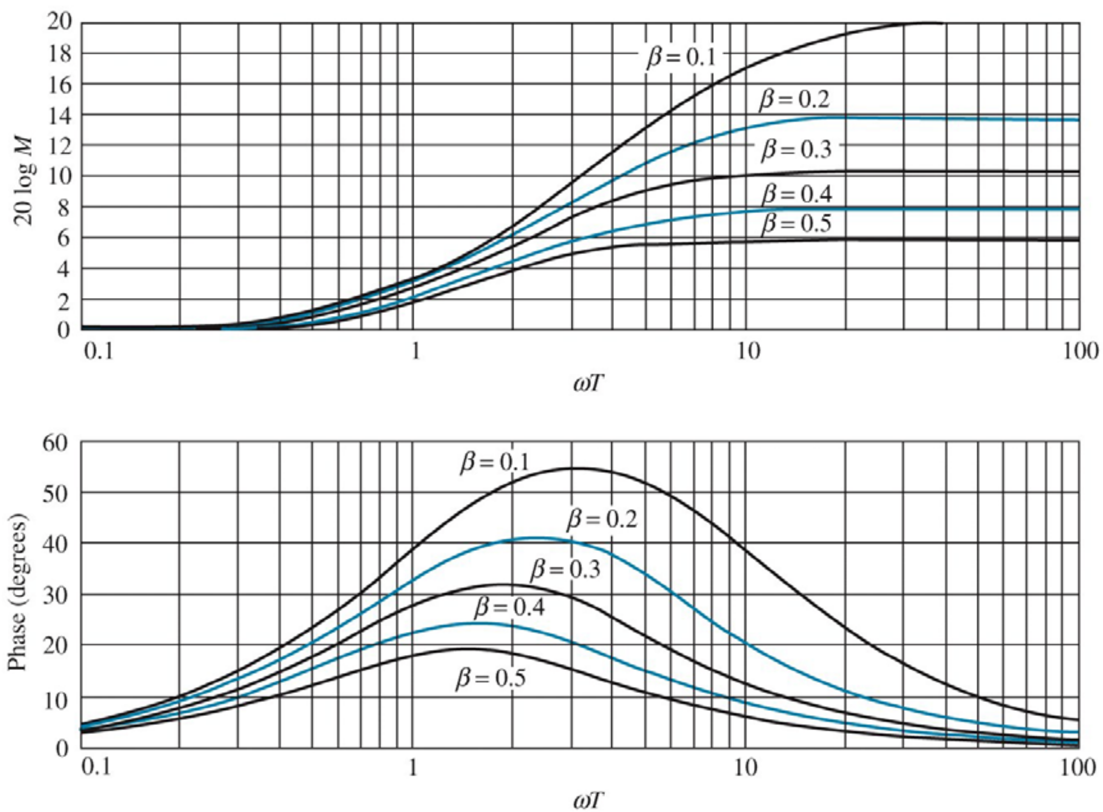


Figure 11.8

- Peaks of the phase curve vary in maximum angle and in the frequency at which the maximum occurs
- The dc gain of the compensator is set to unity with the coefficient *1/β*

Analytical Expression for ϕ_{max} and ω_{max}

$$G_c(s) = \frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} \quad (0<\beta<1)$$

The phase angle of the lead compensator:

$$\phi_c = \tan^{-1}\omega T - \tan^{-1}\omega\beta T$$

Differentiating with respect to *ω*, we obtain

$$\frac{d\phi_c}{d\omega} = \frac{T}{1+(\omega T)^2} - \frac{\beta T}{1+(\omega\beta T)^2}$$

Setting the above equation equal to zero, we find that the frequency, ω_{max}, *at which the* maximum phase angle, ϕ_{max}, occurs is

$$\omega_{\max} = \frac{1}{T\sqrt{\beta}}$$

The maximum phase shift is $$\phi_{\max} = \tan^{-1}\frac{1-\beta}{2\sqrt{\beta}} = \sin^{-1}\frac{1-\beta}{1+\beta}$$

The compensator's magnitude at ω_{max} $$\left|G_c\left(j\omega_{\max}\right)\right| = \frac{1}{\sqrt{\beta}}$$

Example

Example 11.3

Design
D

Lead Compensation Design

PROBLEM: Given the system of Figure 11.2, design a lead compensator to yield a 20% overshoot and $K_v = 40$, with a peak time of 0.1 second.

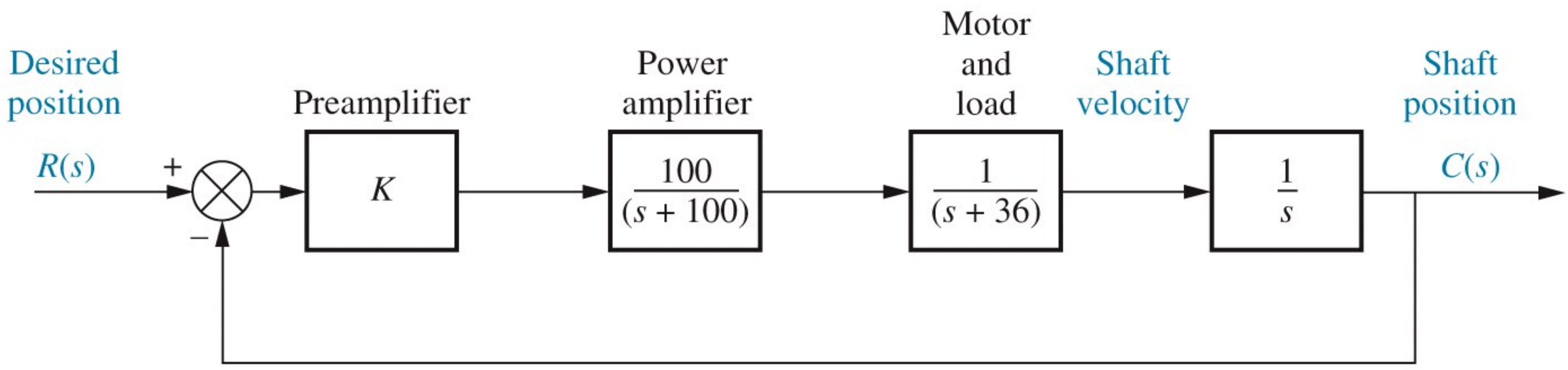


Figure 11.2

SOLUTION: The uncompensated system is $G(s) = 100K/[s(s+36)(s+100)]$. We will follow the outlined procedure.

1. We first look at the closed-loop bandwidth needed to meet the speed requirement imposed by $T_p = 0.1$ second. From Eq. (10.56), with $T_p = 0.1$ second and $\zeta = 0.456$ (i.e., 20% overshoot), a closed-loop bandwidth of 46.6 rad/s is required.

$$\omega_{BW} = \frac{4}{T_p\sqrt{1-\zeta^2}}\sqrt{\left(1-2\zeta^2\right)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

Step 1. Find the closed-loop bandwidth required to meet the settling time, peak time, or rise time requirement (see Eqs. (10.54) through (10.56)).

2. In order to meet the specification of $K_v = 40$, K must be set at 1440, yielding $G(s) = 144{,}000/[s(s+36)(s+100)]$.

$$G(s) = \frac{100K}{s(s+36)(s+100)}$$

Step 2. Since the lead compensator has negligible effect at low frequencies, set the gain, K, of the uncompensated system to the value that satisfies the steady state error requirement.

3. The uncompensated system's frequency response plots for $K = 1440$ are shown in Figure 11.9.

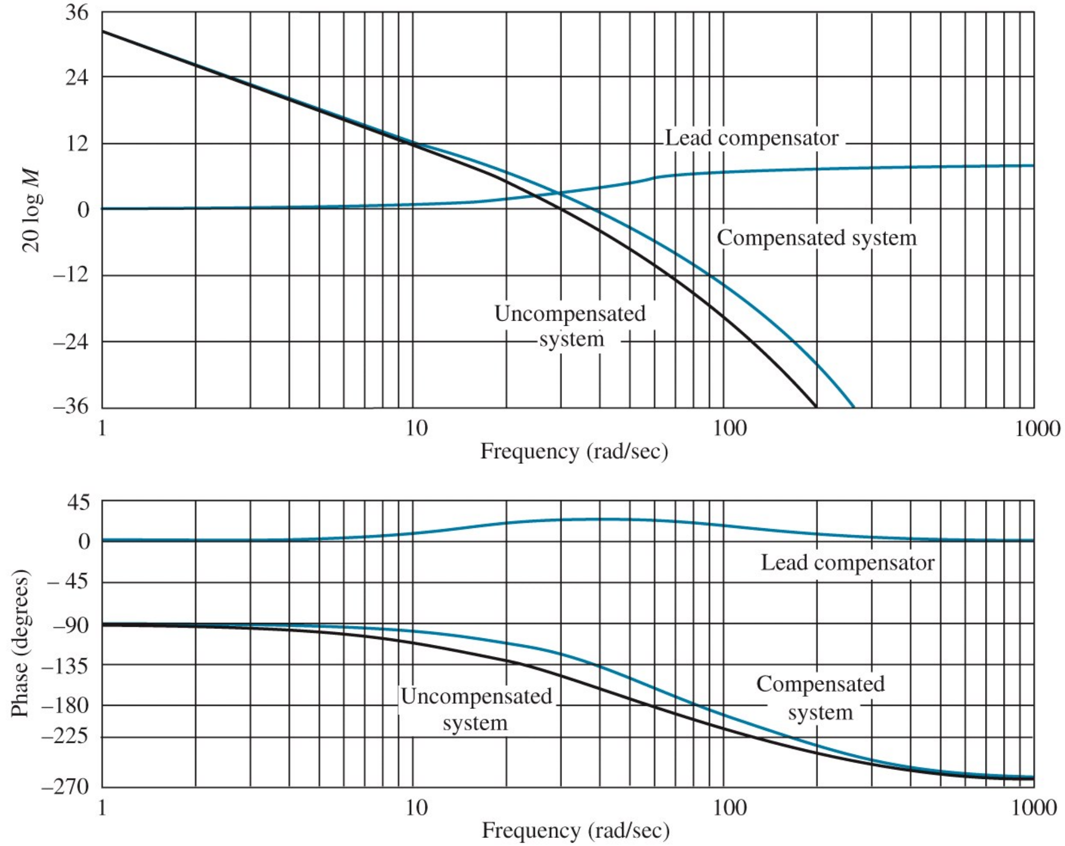


Figure 11.9

Step 3. Plot the Bode magnitude and phase diagrams for this value of gain and determine the uncompensated system's phase margin.

4. A 20% overshoot implies a phase margin of 48.1°. The uncompensated system with $K = 1440$ has a phase margin of 34° at a phase-margin frequency

of 29.6. T … at adds enough
phase to … d network will
also inc … tion factor to
compens … e at this higher
phase-m … phase-margin
frequenc … l phase contri-
bution r … 24.1°. In sum-
mary, ou … of 48.1° with a
bandwid … cceptable after
the desig … y be necessary.

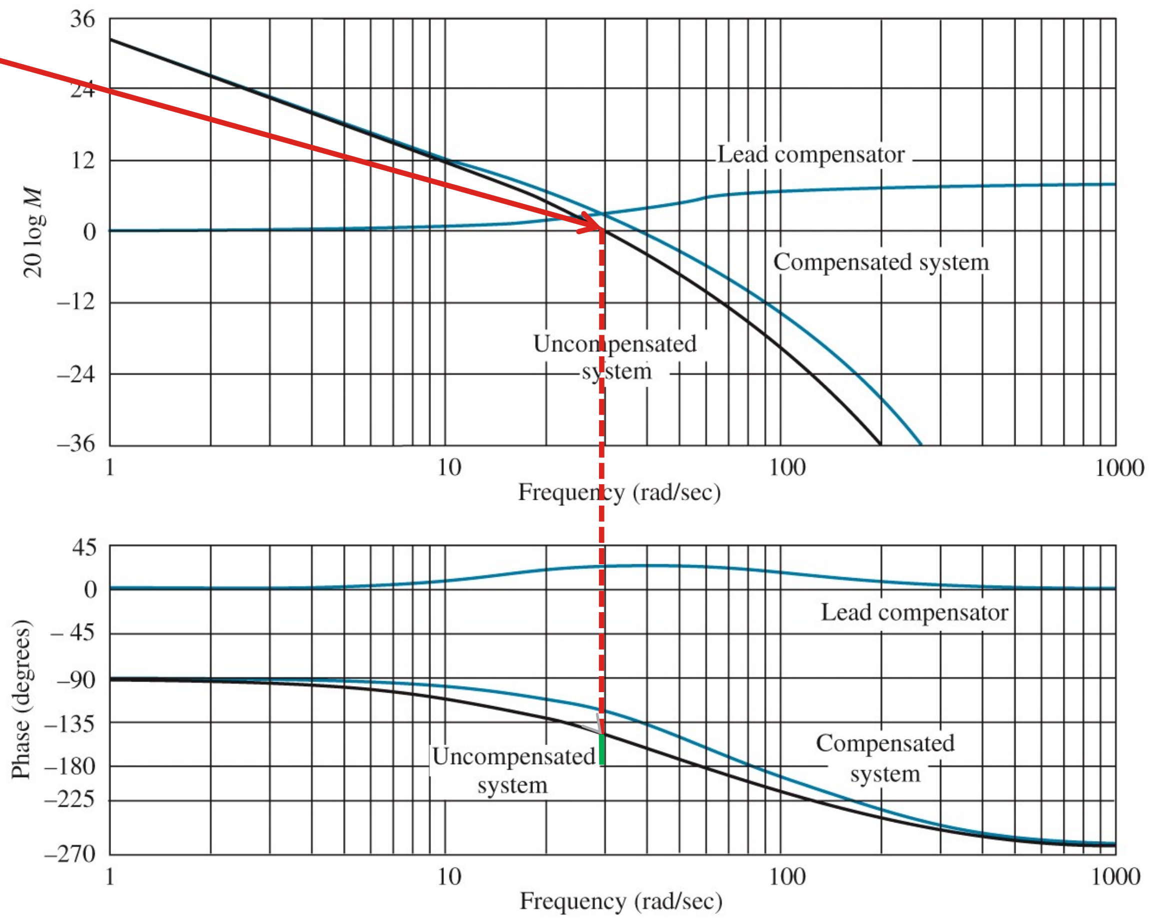


Figure 11.9

Step 4. F … or percent
overshoo … se contribution
required

5. Using Eq. (11.11), $\beta = 0.42$ for $\phi_{max} = 24.1^{\circ}$.

$$\phi_{max} = \tan^{-1}\frac{1-\beta}{2\sqrt{\beta}} = \sin^{-1}\frac{1-\beta}{1+\beta}$$

5. Determine the value of β (see Eqs. (11.6) and (11.11)) from the lead compensator's required phase contribution.

6. From Eq. (11.12), the lead compensator's magnitude is 3.76 dB at ω_{max}.

$$\left|G_c\left(j\omega_{max}\right)\right| = \frac{1}{\sqrt{\beta}}$$

6. Determine the compensator's magnitude at the peak of the phase curve (Eq. (11.12)).

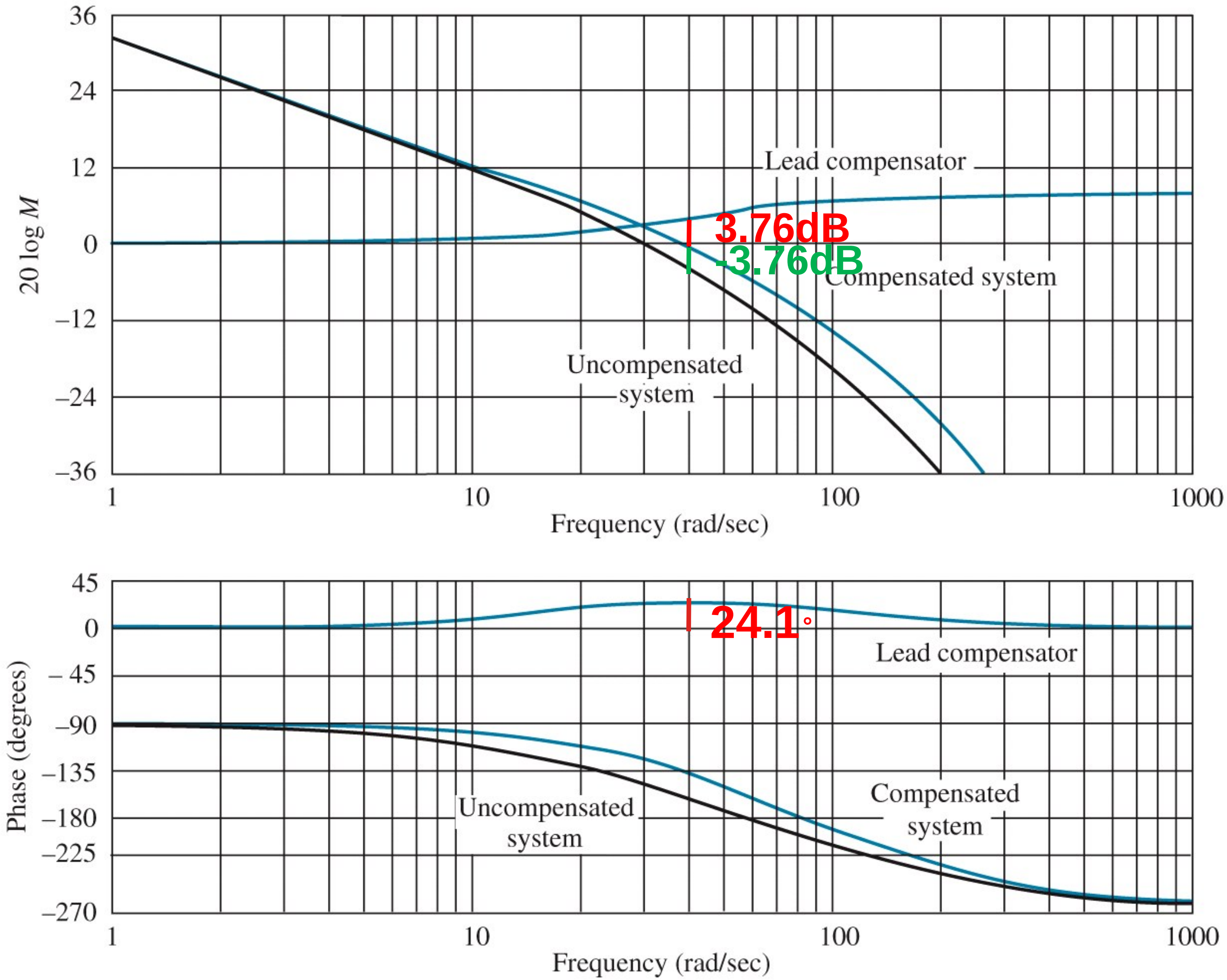

36
24
12
0
–12
–24
–36
20 log M
Lead compensator
3.76dB
-3.76dB
Compensated system
Uncompensated system
1
10
100
1000
Frequency (rad/sec)
45
0
– 45
–90
–135
–180
–225
–270
Phase (degrees)
24.1°
Lead compensator
Uncompensated system
Compensated system
1
10
100
1000
Frequency (rad/sec)

Figure 11.9

7. If we select ω_{max} to be the new phase-margin frequency, the uncompensated system's magnitude at this frequency must be -3.76 dB to yield a 0 dB crossover at ω_{max} for the compensated system. The uncompensated system passes through -3.76 dB at $\omega_{max} = 39$ rad/s. This frequency is thus the new phase-margin frequency.

Step 7. Determine the new phase-margin frequency by finding where the uncompensated system's magnitude curve is the negative of the lead compensator's magnitude at the peak of the compensator's phase curve.

8. We now find the lead compensator's break frequencies. From Eq. (11.9), $1/T = 25.3$ and $1/\beta T = 60.2$.

$$G_c(s) = \frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} \quad (0<\beta<1)$$

$$\omega_{\max} = \frac{1}{T\sqrt{\beta}}$$

Step 8. Design the lead compensator's break frequencies, using Eqs. (11.6) and (11.9) to find T and the break frequencies.

9. Hence, the compensator is given by

$$G_c(s) = \frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} = 2.38\frac{s+25.3}{s+60.2} \tag{11.13}$$

where 2.38 is the gain required to keep the dc gain of the compensator at unity so that $K_v = 40$ after the compensator is inserted.

The final, compensated open-loop transfer function is then

$$G_c(s)G(s) = \frac{342{,}600(s+25.3)}{s(s+36)(s+100)(s+60.2)} \tag{11.14}$$

Step 9. Reset the system gain to compensate for the lead compensator's gain.

10. From ... se is
−7 d ... loop
band ... nent
of 46. ... ısion
abou ... roxi-
matic

Step ... uirement
in Ste

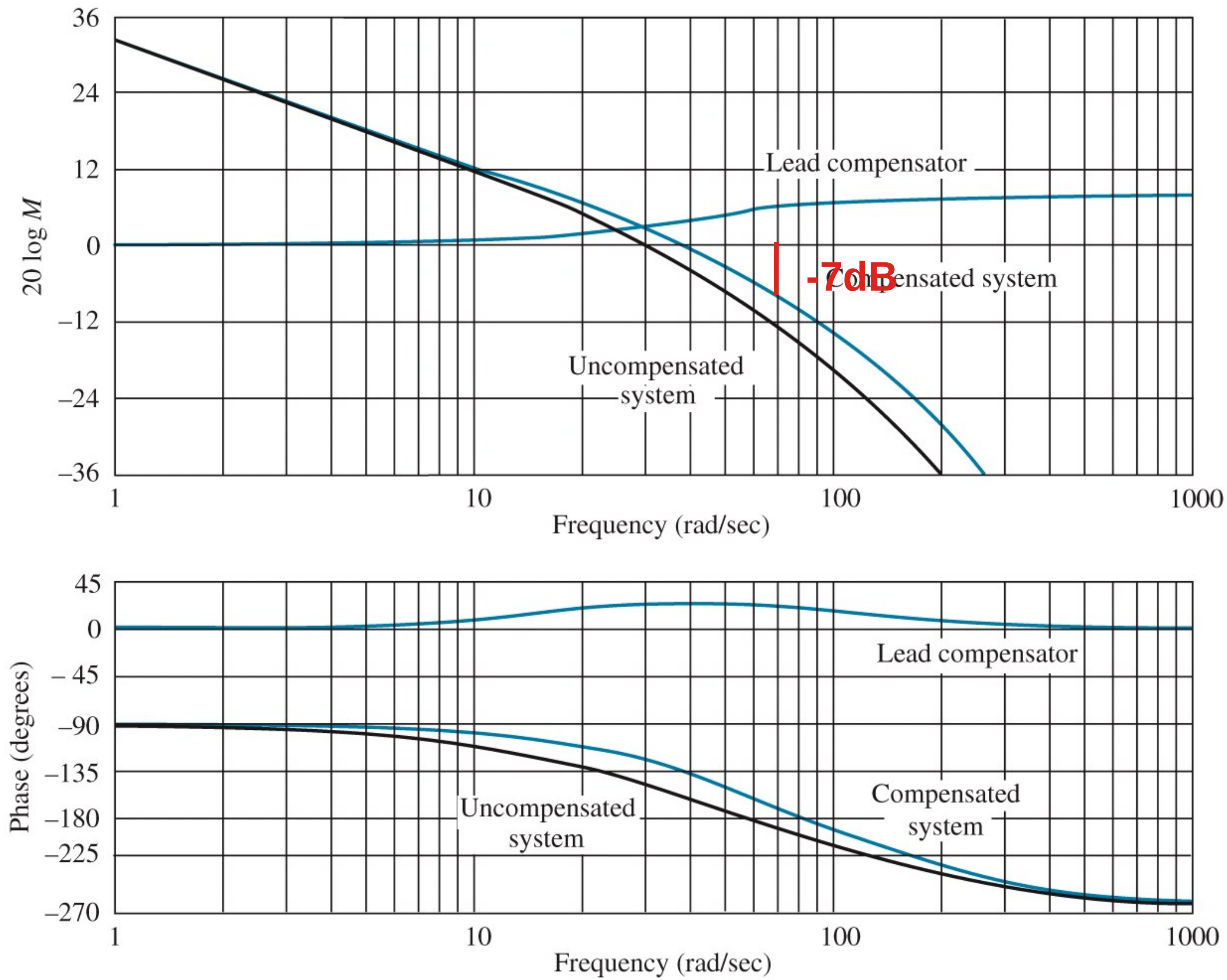


Figure 11.9

11. Figure 11.9 summarizes the design and shows the effect of the compensation. Final results, obtained from a simulation and the actual (nonasymptotic) frequency response, are shown in Table 11.3. Notice the increase in phase margin, phase-margin frequency, and closed-loop bandwidth after the lead compensator was added to the gain-adjusted system. The peak time and the steady-state error requirements have been met, although the phase margin is less than that proposed and the percent overshoot is 2.6% larger than proposed. Finally, if the performance is not acceptable, a redesign is necessary.

TABLE 11.3 Characteristic of the lead-compensated system of Example 11.3

Parameter	Proposed specification	Actual gain-compensated value	Actual lead-compensated value
K_v	40	40	40
Phase margin	48.1°	34°	45.5°
Phase-margin frequency	—	29.6 rad/s	39 rad/s
Closed-loop bandwidth	46.6 rad/s	50 rad/s	68.8 rad/s
Percent overshoot	20	37	22.6
Peak time	0.1 second	0.1 second	0.075 second

Step 11. Simulate to be sure all requirements are met.

Design Procedure

- 1. Find the closed-loop bandwidth required to meet the settling time, peak time, or rise time requirement (see Eqs. (10.54) through (10.56)).
- 2. Since the lead compensator has negligible effect at low frequencies, set the gain, K, of the uncompensated system to the value that satisfies the steady state error requirement.
- 3. Plot the Bode magnitude and phase diagrams for this value of gain and determine the uncompensated system's phase margin.
- 4. Find the phase margin to meet the damping ratio or percent overshoot requirement. Then evaluate the additional phase contribution required from the compensator.[4]

Design Procedure (Cont.)

- 5. Determine the value of β (see Eqs. (11.6) and (11.11)) from the lead compensator's required phase contribution.

$$G_c(s)=\frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}}\quad(0<\beta<1)$$

$$\phi_{\max}=\tan^{-1}\frac{1-\beta}{2\sqrt{\beta}}=\sin^{-1}\frac{1-\beta}{1+\beta}$$

- 6. Determine the compensator's magnitude at the peak of the phase curve (Eq. (11.12)).

$$\left|G_c(j\omega_{\max})\right|=\frac{1}{\sqrt{\beta}}$$

Design Procedure (Cont.)

- 7. Determine the new phase-margin frequency by finding where the uncompensated system's magnitude curve is the negative of the lead compensator's magnitude at the peak of the compensator's phase curve.
- 8. Design the lead compensator's break frequencies, using Eqs. (11.6) and (11.9) to find T and the break frequencies.

$$G_c(s)=\frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} \quad (0<\beta<1)$$

$$\omega_{\max}=\frac{1}{T\sqrt{\beta}}$$

Design Procedure (Cont.)

- 9. Reset the system gain to compensate for the lead compensator’s gain.
- 10. Check the bandwidth to be sure the speed requirement in Step 1 has been met.
- 11. Simulate to be sure all requirements are met.
- 12. Redesign if necessary to meet requirements.
- From these steps, we see that we are increasing both the amount of phase margin (improving percent overshoot) and the gain crossover frequency (increasing the speed).

Skill-Assessment Exercise 11.3

WileyPLUS
WPCS
Control Solutions

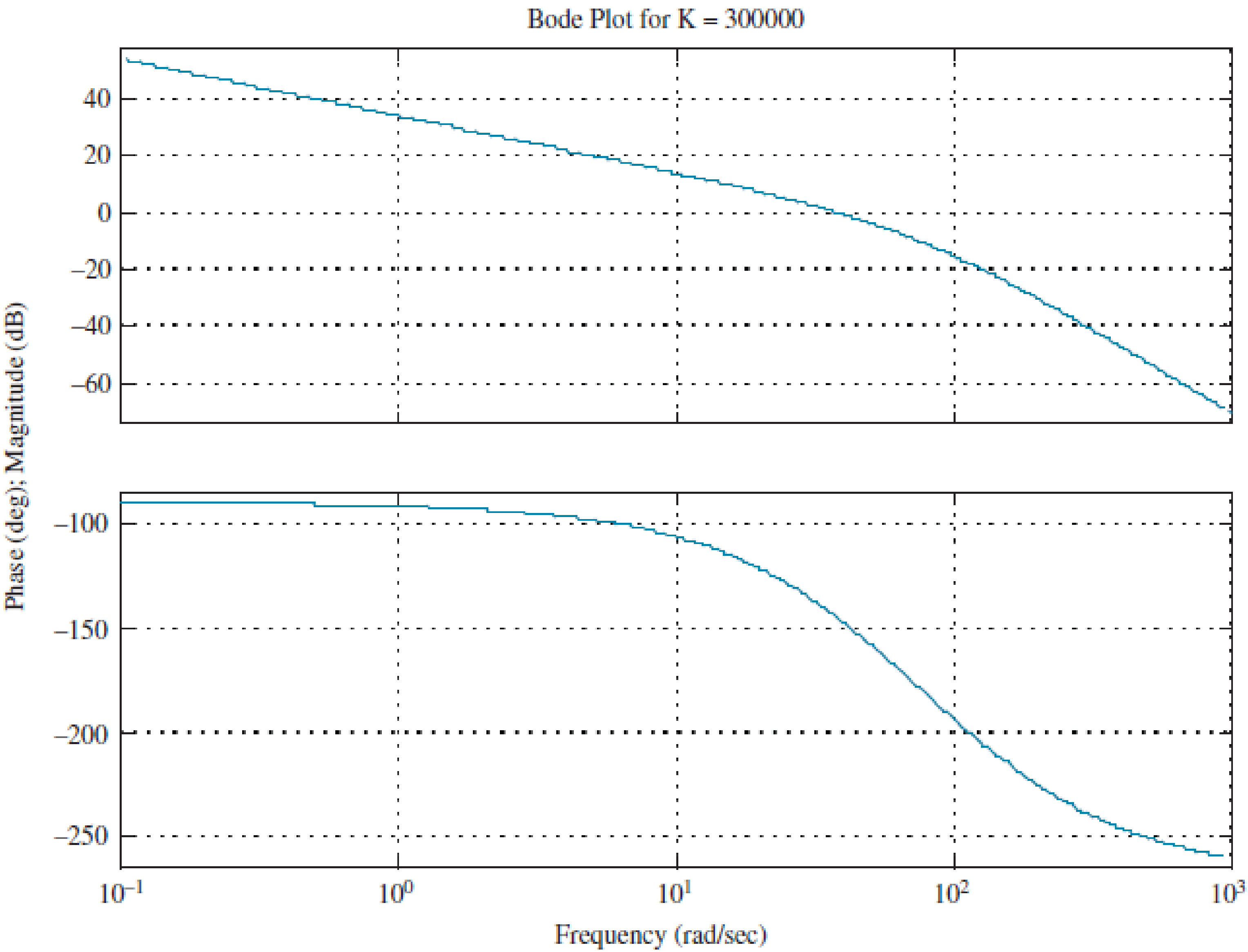
PROBLEM: Design a lead compensator for the system in Skill-Assessment Exercise 11.1 to meet the following specifications: $\%OS = 20\%$, $T_s = 0.2$ s and $K_v = 50$.

ANSWER: $G_{lead}(s) = \dfrac{2.27(s+33.2)}{(s+75.4)}$; $G(s) = \dfrac{300{,}000}{s(s+50)(s+120)}$

The complete solution is at www.wiley.com/college/nise.

11.3

A 20% overshoot requires $\zeta = \dfrac{-\log\left(\frac{\%}{100}\right)}{\sqrt{\pi^2 + \log^2\left(\frac{\%}{100}\right)}} = 0.456$. The required bandwidth is then calculated as $\omega_{BW} = \dfrac{4}{T_s\zeta}\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} = 57.9$ rad/s. In order to meet the steady-state error requirement of $K_v = 50 = \dfrac{K}{(50)(120)}$, we calculate $K = 300,000$. The uncompensated Bode plot for this gain is shown below.


Bode Plot for K = 300000
Phase (deg); Magnitude (dB)
40
20
0
–20
–40
–60
–100
–150
–200
–250
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)

The uncompensated system's phase margin measurement is taken where the magnitude plot crosses 0 dB. We find that when the magnitude plot crosses 0 dB, the phase angle is -144.8°. Therefore, the uncompensated system's phase margin is $-180^\circ + 144.8^\circ = 35.2^\circ$. The required phase margin based on the required damping ratio is $\Phi_M = \tan^{-1}\dfrac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}} = 48.1^\circ$. Adding a 10° correction factor, the required phase margin is 58.1°. Hence, the compensator must contribute $\phi_{\max} = 58.1^\circ - 35.2^\circ = 22.9^\circ$. Using $\phi_{\max} = \sin^{-1}\dfrac{1-\beta}{1+\beta}$, $\beta = \dfrac{1-\sin\phi_{\max}}{1+\sin\phi_{\max}} = 0.44$. The compensator's peak magnitude is calculated as $M_{\max} = \dfrac{1}{\sqrt{\beta}} = 1.51$. Now find the frequency at which the uncompensated system has a magnitude $1/M_{\max}$, or -3.58 dB. From the Bode plot, this magnitude occurs at $\omega_{\max} = 50\,\text{rad/s}$. The compensator's zero is at $z_c = \dfrac{1}{T}$. $\omega_{\max} = \dfrac{1}{T\sqrt{\beta}}$ Therefore, $z_c = 33.2$.

The compensator's pole is at $P_c = \dfrac{1}{\beta T} = \dfrac{z_c}{\beta} = 75.4$. The compensator gain is chosen to yield unity gain at dc.

Hence, $K_c = 75.4/33.2 = 2.27$. Summarizing, $G_c(s) = 2.27\dfrac{(s+33.2)}{(s+75.4)}$, and $G(s)\dfrac{300{,}000}{s(s+50)(s+120)}$.

5. Lag-Lead Compensation

- Lag compensation
  - Lag compensation consists of setting the gain to meet the steady-state error requirement and then reducing the high-frequency gain to create stability

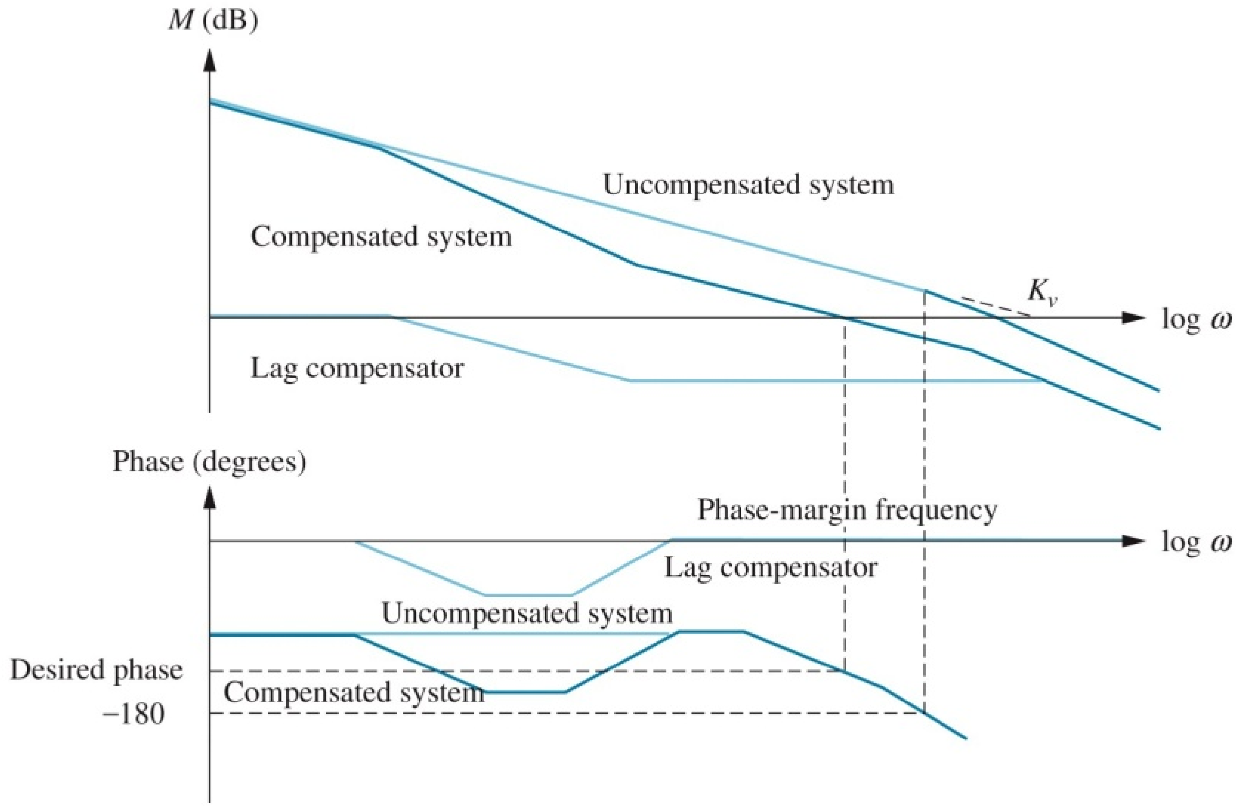


Figure 11.4

- Lead compensation
  - The lead compensator increases the phase angle at high frequencies. The effect is to produce a faster system.

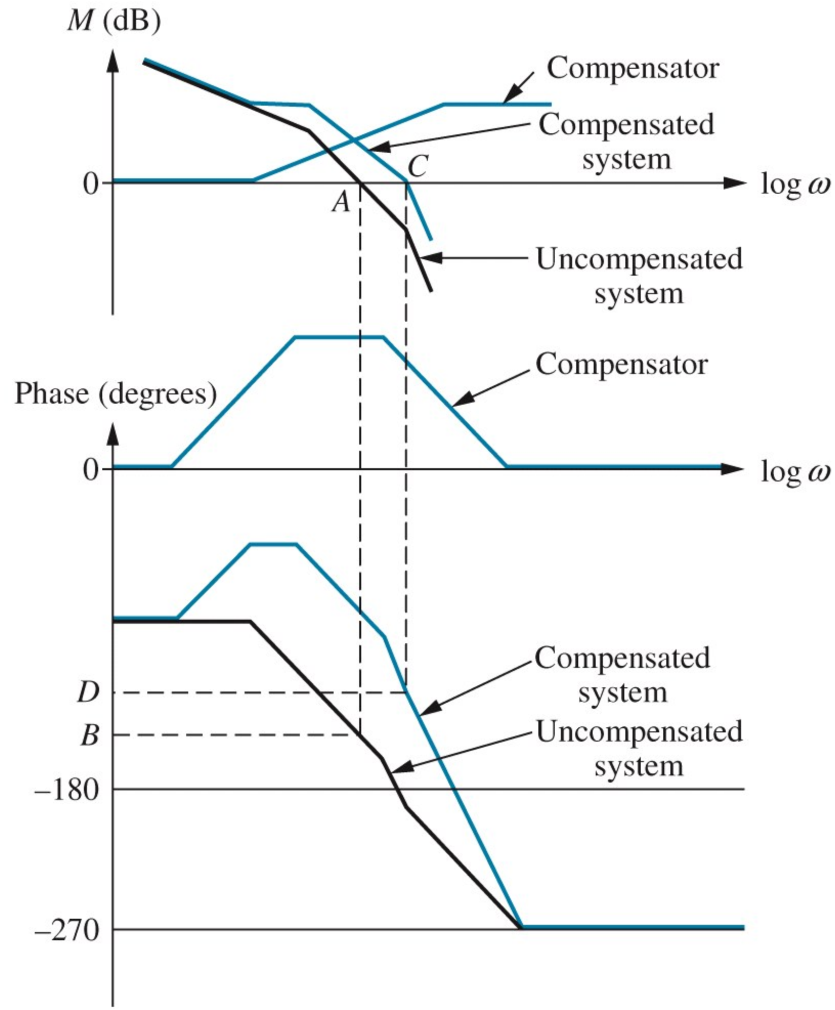


Figure 11.7

- A lag compensator is basically a low-pass filter. The low-frequency gain can be raised to improve the steady-state error, and the high-frequency gain is reduced to yield stability.
- A lead compensator is basically a high-pass filter. The lead compensator increases the high-frequency gain while keeping the low-frequency gain the same. the lead compensator increases the phase angle at high frequencies. The effect is to produce a faster, stable system since the uncompensated phase margin now occurs at a higher frequency.
- A lag-lead compensator combines the advantages of both the lag and the lead compensator.

Lag-Lead Compensation

- Lag-lead compensator: A transfer function, characterized by a pole-zero configuration that is the combination of a lag and a lead compensator, that is used for the purpose of improving both the transient response and the steady-state error of a closed-loop system.
- Design method:
  - Design the lag compensation to lower the high-frequency gain, stabilize the system, and improve the steady-state error;
  - Design a lead compensator to meet the phase-margin requirements.

Lag-Lead Compensation

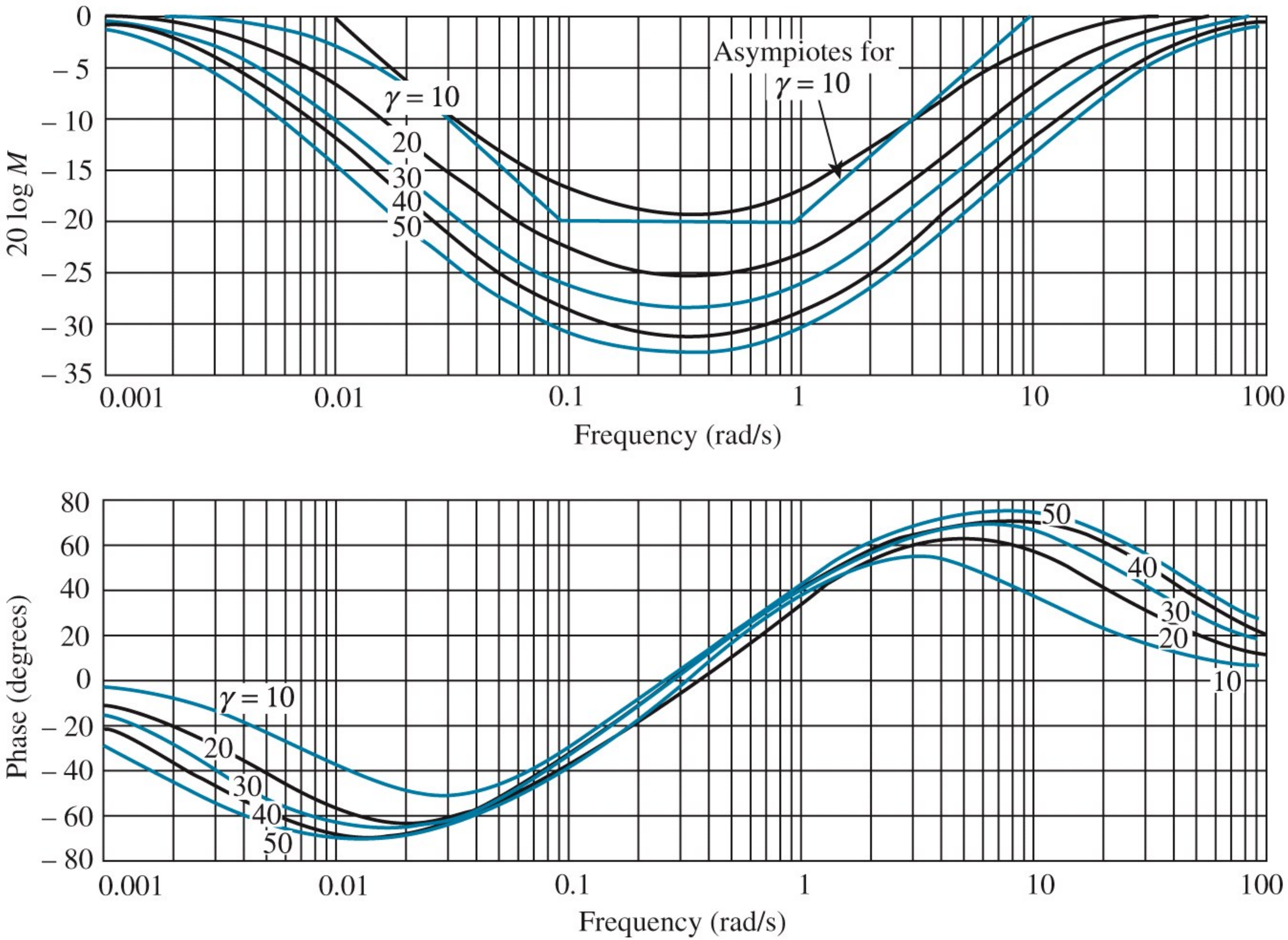
$$G_{\text{lead}}(s) = \frac{1}{\beta}\frac{s+\frac{1}{T}}{s+\frac{1}{\beta T}} \quad (0<\beta<1)$$

$$G_{\text{lag}}(s) = \frac{s+\frac{1}{T}}{s+\frac{1}{\alpha T}} \quad (\alpha>1)$$

Change the phase diagram, increasing the phase margin to reduce the percent overshoot, and increasing the gain crossover to realize a faster transient response

Reduce the high-frequency gain to improve the steady-state error and create stability.

$$G_c(s) = G_{\text{Lead}}(s)G_{\text{lag}}(s) = \left(\frac{s+\frac{1}{T_1}}{s+\frac{\gamma}{T_1}}\right)\left(\frac{s+\frac{1}{T_2}}{s+\frac{1}{\gamma T_2}}\right), \quad (\gamma>1)$$



0
– 5
– 10
– 15
– 20
– 25
– 30
– 35
20 log M
γ = 10
20
30
40
50
Asympiotes for
γ = 10
0.001
0.01
0.1
1
10
100
Frequency (rad/s)
80
60
40
20
0
– 20
– 40
– 60
– 80
Phase (degrees)
γ = 10
20
30
40
50
10
Frequency (rad/s)

Figure 11.11

The National Advanced Driving Simulator

(*a*)

Figure 11.10a

(*b*)

Figure 11.10b

Katharina Bosse/laif/Redux Pictures

Example

Example 11.4

Lag-Lead Compensation Design

PROBLEM: Given a unity feedback system where $G(s) = K/[s(s+1)(s+4)]$, design a passive lag-lead compensator using Bode diagrams to yield a 13.25% overshoot, a peak time of 2 seconds, and $K_v = 12$.

1. The bandwidth required for a 2-seconds peak time is

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

$$\omega_{BW} = \frac{4}{T_s \zeta}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

$$\omega_{BW} = \frac{4}{T_p\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2)+\sqrt{4\zeta^4-4\zeta^2+2}}$$

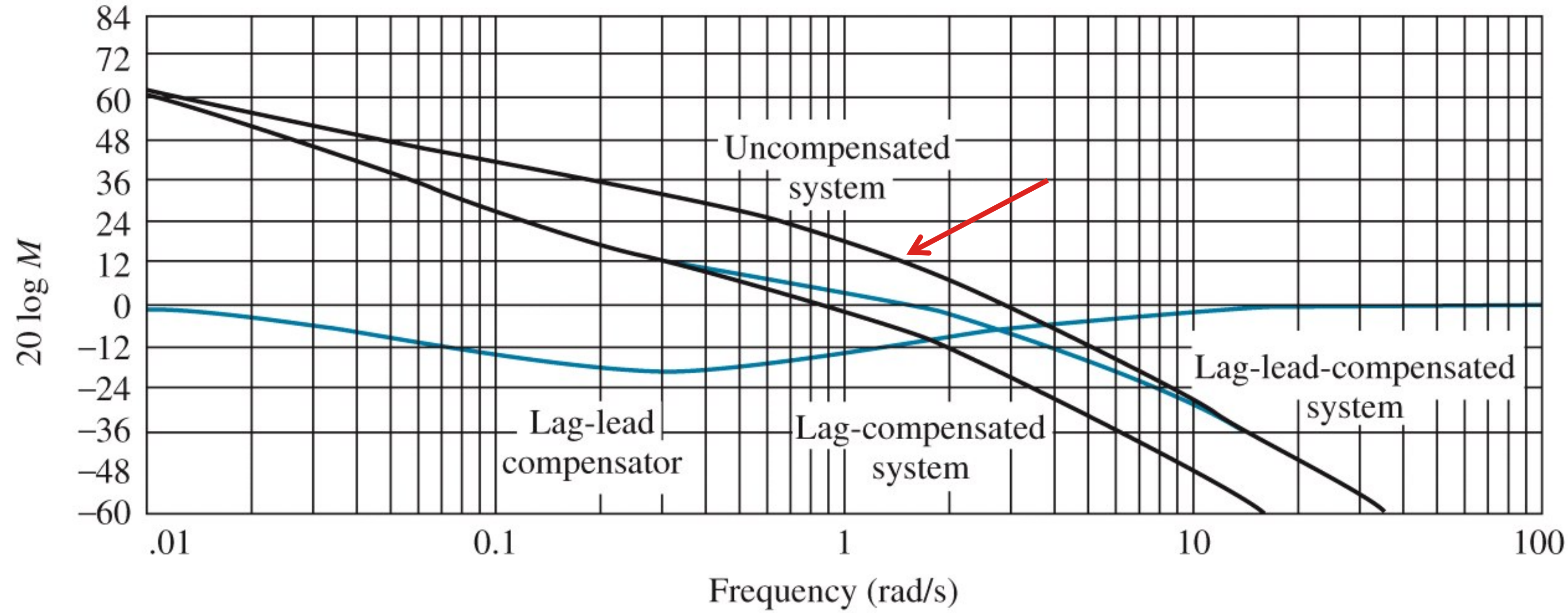
Step 1. Using a second-order approximation, find the closed-loop bandwidth required to meet the settling time, peak time, or rise time requirement (see Eqs. (10.55) and (10.56)).

2. In order to meet the steady-state error requirement, $K_v = 12$, the value of K is 48.

Step 2. Set the gain, *K*, to the value required by the steady-state error specification.

3. The Bode plots for the uncompensated system with $K = 48$ are shown in Figure 11.12. We can see that the system is unstable.

Step 3. Plot the Bode magnitude and phase diagrams for this value of gain.


84
72
60
48
36
24
12
0
–12
–24
–36
–48
–60
20 log M
Uncompensated system
Lag-lead-compensated system
Lag-lead compensator
Lag-compensated system
.01
0.1
1
10
100
Frequency (rad/s)

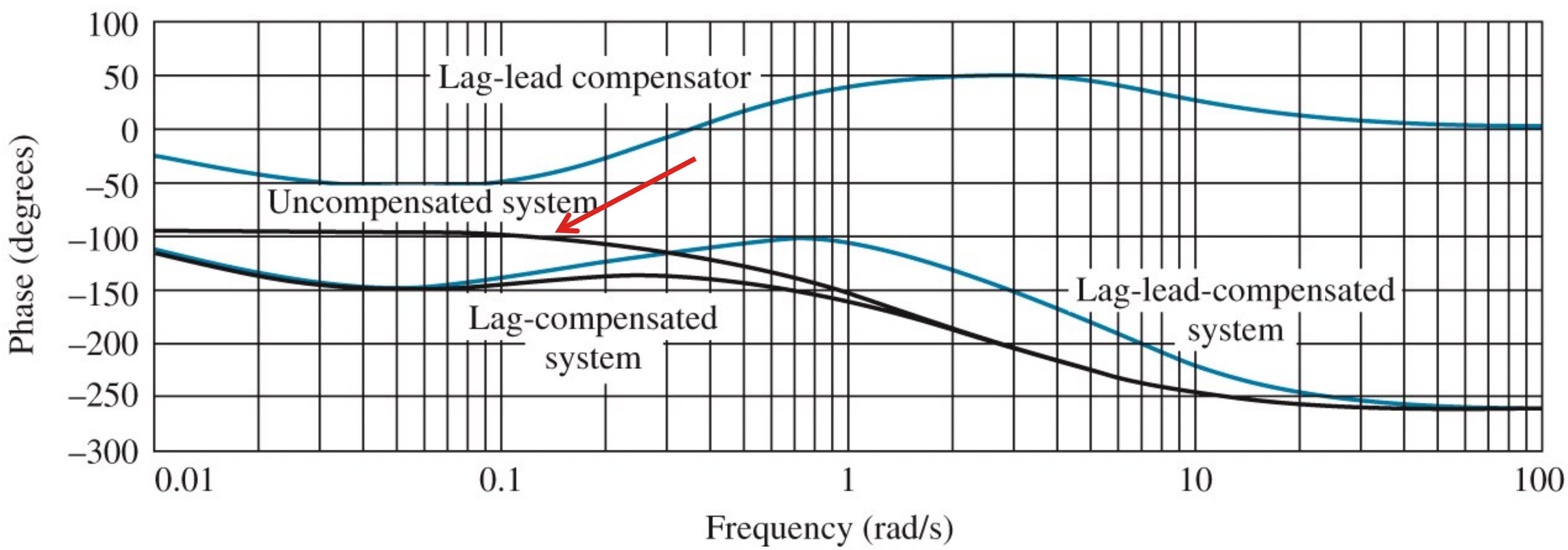

100
50
0
–50
–100
–150
–200
–250
–300
Phase (degrees)
Lag-lead compensator
Uncompensated system
Lag-compensated system
Lag-lead-compensated system
0.01
0.1
1
10
100
Frequency (rad/s)

Figure 11.12

4. The required phase margin to yield a 13.25% overshoot is 55°.

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

$$\Phi_M = 90 - \tan^{-1}\frac{\omega_1}{2\zeta\omega_n} = \tan^{-1}\frac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}}$$

Step 4. Using a second-order approximation, calculate the phase margin to meet the damping ratio or percent overshoot requirement, using Eq. (10.73).

5. Let us select $\omega = 1.8$ rad/s as the new phase-margin frequency.

Step 5. Select a new phase-margin frequency near ω_{BW}.

($\omega=0.8*\omega_{BW}$)

6. At this frequency, the uncompensated phase is −176° and would require, if we add a −5° contribution from the lag compensator, a 56° contribution from the lead portion of the compensator.

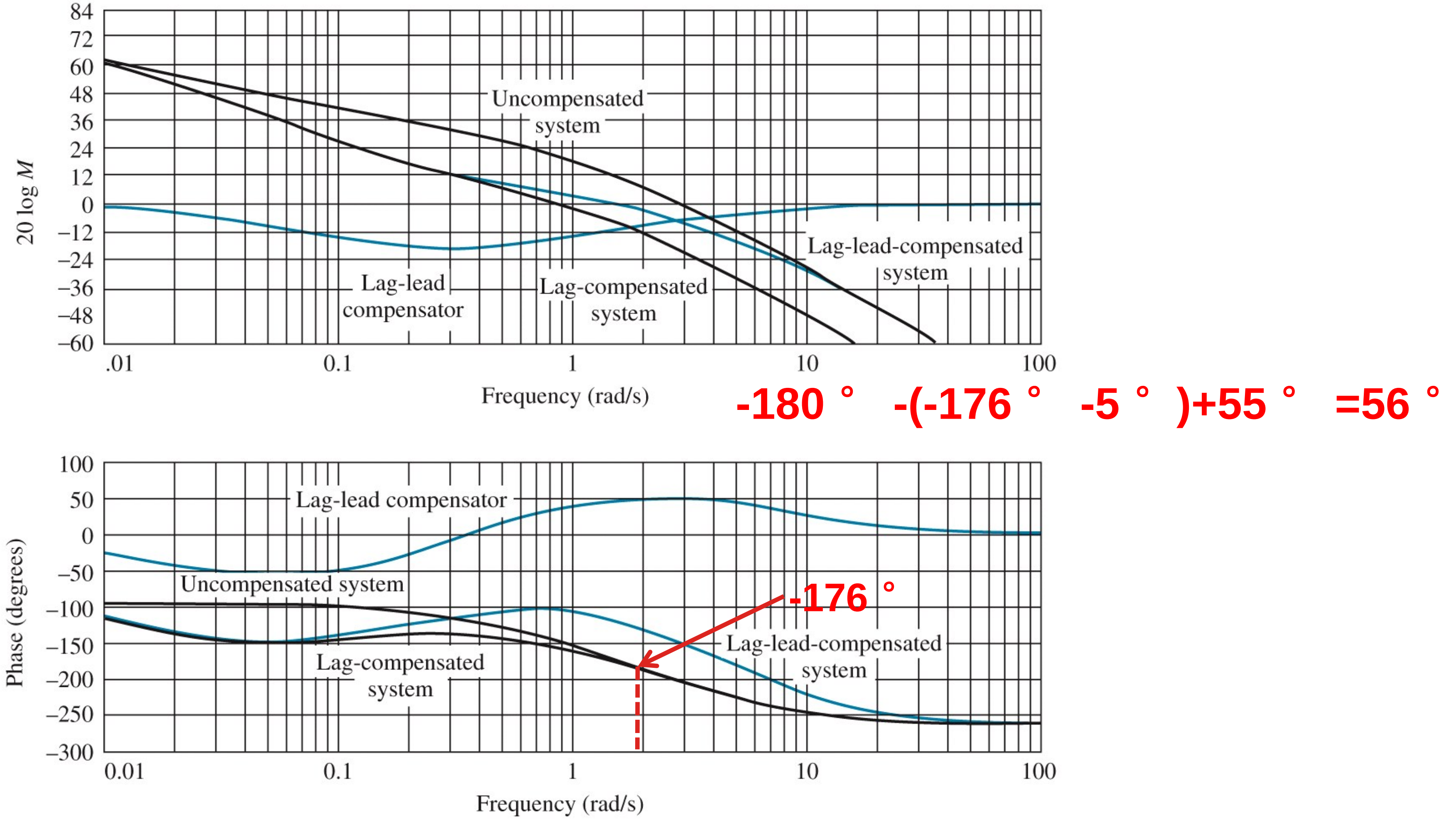


Figure 11.12

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Step 6. At the new phase-margin frequency, determine the additional amount of phase lead required to meet the phase-margin requirement. Add a small contribution that will be required after the addition of the lag compensator.

7. The design of the lag compensator is next. The lag compensator allows us to keep the gain of 48 required for $K_v = 12$ and not have to lower the gain to stabilize the system. As long as the lag compensator stabilizes the system, the design parameters are not critical since the phase margin will be designed with the lead compensator. Thus, choose the lag compensator so that its phase response will have minimal effect at the new phase-margin frequency. Let us choose the lag compensator's higher break frequency to be 1 decade below the new phase-margin frequency, at 0.18 rad/s. Since we need to add 56° of phase shift with the lead compensator at $\omega = 1.8$ rad/s, we estimate from Figure 11.8 that, if $\gamma = 10.6$ (since $\gamma = 1/\beta$, $\beta = 0.094$), we can obtain about 56° of phase shift from the lead compensator. Thus with $\gamma = 10.6$ and a new phase-margin frequency of $\omega = 1.8$ rad/s, the transfer function of the lag compensator is

$$G_{\text{lag}}(s) = \frac{1}{\gamma}\frac{\left(s + \dfrac{1}{T_2}\right)}{\left(s + \dfrac{1}{\gamma T_2}\right)} = \frac{1}{10.6}\frac{(s + 0.183)}{(s + 0.0172)} \qquad (11.16)$$

where the gain term, $1/\gamma$, keeps the dc gain of the lag compensator at 0 dB. The lag-compensated system's open-loop transfer function is

$$G_{\text{lag-comp}}(s) = \frac{4.53(s + 0.183)}{s(s+1)(s+4)(s+0.0172)} \qquad (11.17)$$

- Step 7. Design the lag compensator by selecting the higher break frequency one decade below the new phase-margin frequency. The design of the lag compensator is not critical, and any design for the proper phase margin will be relegated to the lead compensator. The lag compensator simply provides stabilization of the system with the gain required for the steady-state error specification. Find the value of γ from the lead compensator's requirements. Using the phase required from the lead compensator, the phase response curve of Figure 11.8 can be used to find the value of $\gamma=1/\beta$. This value, along with the previously found lag's upper break frequency, allows us to find the lag's lower break frequency.

8. Now we design the lead compensator. At $\omega = 1.8$, the lag-compensated system has a phase angle of $180°$. Using the values of $\omega_{\max} = 1.8$ and $\beta = 0.094$, Eq. (11.9) yields the lower break, $1/T_1 = 0.56$ rad/s. The higher break is then $1/\beta T_1 = 5.96$ rad/s. The lead compensator is

$$G_{\text{lead}}(s) = \gamma \frac{\left(s + \frac{1}{T_1}\right)}{\left(s + \frac{\gamma}{T_1}\right)} = 10.6 \frac{(s + 0.56)}{(s + 5.96)} \tag{11.18}$$

The lag-lead-compensated system's open-loop transfer function is

$$G_{\text{lag-lead-comp}}(s) = \frac{48(s + 0.183)(s + 0.56)}{s(s + 1)(s + 4)(s + 0.0172)(s + 5.96)} \tag{11.19}$$

Step 8. Design the lead compensator. Using the value of *γ* from the lag compensator design and the value assumed for the new phase-margin frequency, find the lower and upper break frequency for the lead compensator, using Eq. (11.9) and solving for *T*.

9. Now check the bandwidth. The closed-loop bandwidth is equal to that frequency where the open-loop magnitude response is approximately –7 dB. From Figure 11.12, the magnitude is –7 dB at approximately 3 rad/s. This bandwidth exceeds that required to meet the peak time requirement.

The design is now checked with a simulation to obtain actual performance values. Table 11.4 summarizes the system's characteristics. The peak time requirement is also met. Again, if the requirements were not met, a redesign would be necessary.

TABLE 11.4 Characteristics of gain-compensated system of Example 11.4

Parameter	Proposed specification	Actual value
K_v	12	12
Phase margin	55°	59.3°
Phase-margin frequency	—	1.63 rad/s
Closed-loop bandwidth	2.29 rad/s	3 rad/s
Percent overshoot	13.25	10.2
Peak time	2.0 seconds	1.61 seconds

Step 9. Check the bandwidth to be sure the speed requirement in Step 1 has been met.

Design Procedure

- 1. Using a second-order approximation, find the closed-loop bandwidth required to meet the settling time, peak time, or rise time requirement (see Eqs. (10.55) and (10.56)).
- 2. Set the gain, *K*, to the value required by the steady-state error specification.
- 3. Plot the Bode magnitude and phase diagrams for this value of gain.
- 4. Using a second-order approximation, calculate the phase margin to meet the damping ratio or percent overshoot requirement, using Eq. (10.73).
- 5. Select a new phase-margin frequency near ω_{BW}. ($0.8^* \omega_{BW}$)

- 6. At the new phase-margin frequency, determine the additional amount of phase lead required to meet the phase-margin requirement. Add a small contribution that will be required after the addition of the lag compensator.

- 7. Design the lag compensator by selecting the higher break frequency one decade below the new phase-margin frequency. The design of the lag compensator is not critical, and any design for the proper phase margin will be relegated to the lead compensator. The lag compensator simply provides stabilization of the system with the gain required for the steady-state error specification. Find the value of γ from the lead compensator's requirements. Using the phase required from the lead compensator, the phase response curve of Figure 11.8 can be used to find the value of $\gamma=1/\beta$. This value, along with the previously found lag's upper break frequency, allows us to find the lag's lower break frequency.

- 8. Design the lead compensator. Using the value of γ from the lag compensator design and the value assumed for the new phase-margin frequency, find the lower and upper break frequency for the lead compensator, using Eq. (11.9) and solving for T.
- 9. Check the bandwidth to be sure the speed requirement in Step 1 has been met.
- 10. Redesign if phase-margin or transient specifications are not met, as shown by analysis or simulation.

Skill-Assessment Exercise 11.4

PROBLEM: Design a lag-lead compensator for a unity feedback system with the forward-path transfer function

$$G(s) = \frac{K}{s(s+8)(s+30)}$$

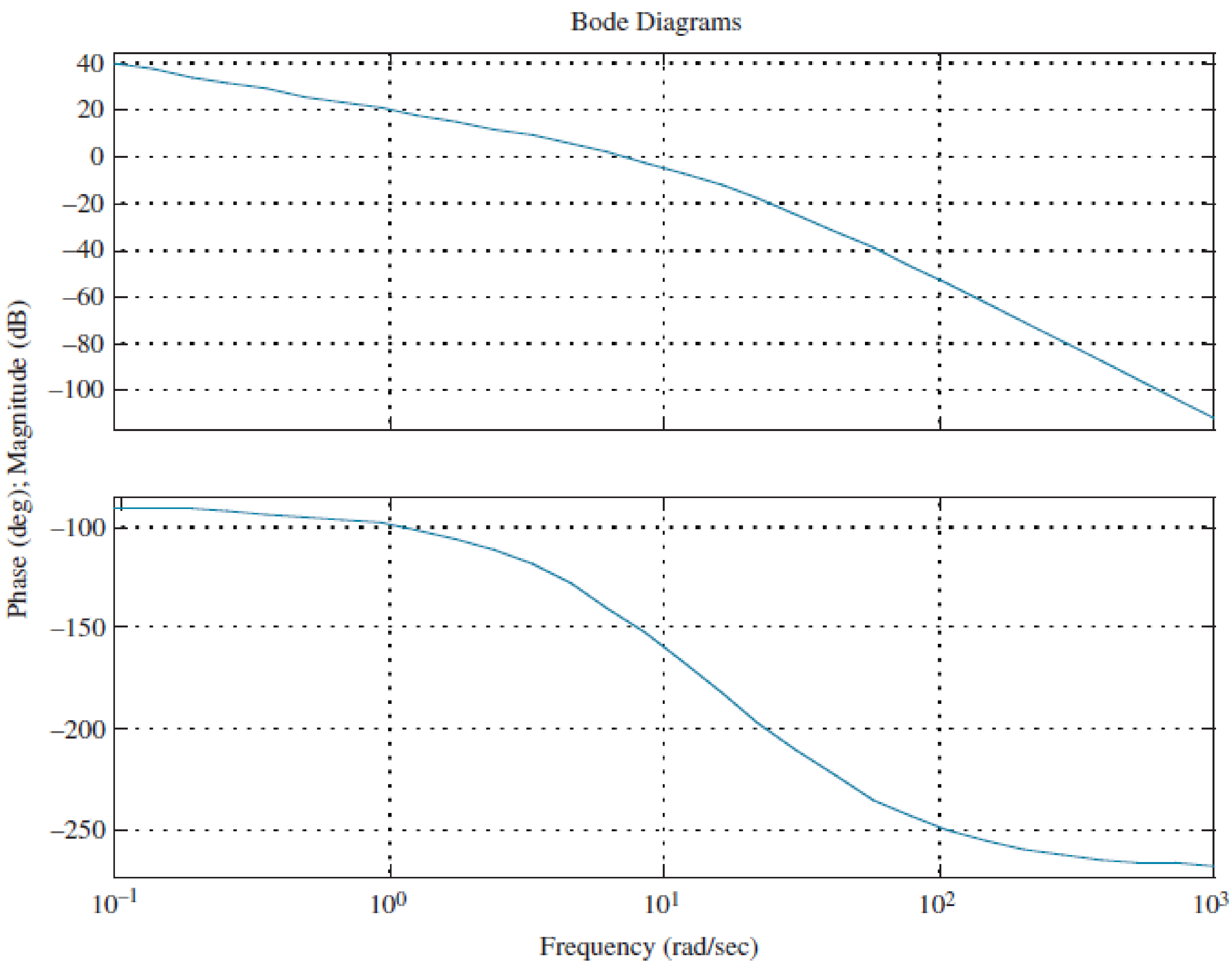
to meet the following specifications: $\%OS = 10\%$, $T_p = 0.6$ s, and $K_v = 10$. Use frequency response techniques.

ANSWER: $G_{\text{lag}}(s) = 0.456\frac{(s+0.602)}{(s+0.275)}$; $G_{\text{lead}}(s) = 2.19\frac{(s+4.07)}{(s+8.93)}$; $K = 2400$.

The complete solution is at www.wiley.com/college/nise.

A 10% overshoot requires $\zeta = \dfrac{-\log\left(\frac{\%}{100}\right)}{\sqrt{\pi^2 + \log^2\left(\frac{\%}{100}\right)}} = 0.591$. The required bandwidth is then calculated as $\omega_{BW} = \dfrac{\pi}{T_p\sqrt{1-\zeta^2}}\sqrt{(1-2\zeta^2) + \sqrt{4\zeta^4 - 4\zeta^2 + 2}} = 7.53\ \mathrm{rad/s}$.

In order to meet the steady-state error requirement of $K_v = 10 = \dfrac{K}{(8)(30)}$, we calculate $K = 2400$. The uncompensated Bode plot for this gain is shown below.


Bode Diagrams
40
20
0
–20
–40
–60
–80
–100
Phase (deg); Magnitude (dB)
–100
–150
–200
–250
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)

Let us select a new phase-margin frequency at $0.8\omega_{BW} = 6.02\,\text{rad/s}$. The required phase margin based on the required damping ratio is $\Phi_M = \tan^{-1}\dfrac{2\zeta}{\sqrt{-2\zeta^2 + \sqrt{1+4\zeta^4}}} = 58.6^\circ$. Adding a 5° correction factor, the required phase margin is 63.6°. At 6.02 rad/s, the new phase-margin frequency, the phase angle is–which represents a phase margin of $180^\circ - 138.3^\circ = 41.7^\circ$. Thus, the lead compensator must contribute $\phi_{\max} = 63.6^\circ - 41.7^\circ = 21.9^\circ$.

Using $\phi_{\max} = \sin^{-1}\dfrac{1-\beta}{1+\beta}$, $\beta = \dfrac{1-\sin\phi_{\max}}{1+\sin\phi_{\max}} = 0.456$.

We now design the lag compensator by first choosing its higher break frequency one decade below the new phase-margin frequency, that is, $z_{lag} = 0.602\,\text{rad/s}$. The lag compensator's pole is $p_{lag} = \beta z_{lag} = 0.275$. Finally, the lag compensator's gain is $K_{lag} = \beta = 0.456$.

Now we design the lead compensator. The lead zero is the product of the new phase margin frequency and $\sqrt{\beta}$, or $z_{lead} = 0.8\omega_{BW}\sqrt{\beta} = 4.07$. Also, $p_{lead} = \frac{z_{lead}}{\beta} = 8.93$. Finally, $K_{lead} = \frac{1}{\beta} = 2.19$. Summarizing,

$$G_{lag} = (s) = 0.456\frac{(s+0.602)}{(s+0.275)};\ G_{lead}(s) = 2.19\frac{(s+4.07)}{(s+8.93)}; \quad \text{and } k = 2400.$$

Summary

- We learned how to design by gain adjustment as well as cascaded lag, lead, and lag-lead compensation.
- Time response characteristics were related to the phase margin, phase-margin frequency, and bandwidth.
- Design by gain adjustment consisted of adjusting the gain to meet a phase margin specification. We located the phase-margin frequency and adjusted the gain to 0 dB.
- A lag compensator is basically a low-pass filter. The low-frequency gain can be raised to improve the steady-state error, and the high-frequency gain is reduced to yield stability. Lag compensation consists of setting the gain to meet the steady-state error requirement and then reducing the high-frequency gain to create stability and meet the phase-margin requirement for the transient response.

Summary (Cont.)

- A lead compensator is basically a high-pass filter. The lead compensator increases the high-frequency gain while keeping the low-frequency gain the same. Thus, the steady-state error can be designed first. At the same time, the lead compensator increases the phase angle at high frequencies. The effect is to produce a faster, stable system since the uncompensated phase margin now occurs at a higher frequency.

- A lag-lead compensator combines the advantages of both the lag and the lead compensator. First, the lag compensator is designed to yield the proper steady-state error with improved stability. Next, the lead compensator is designed to speed up the transient response. If a single network is used as the lag-lead, additional design considerations are applied so that the ratio of the lag zero to the lag pole is the same as the ratio of the lead pole to the lead zero.