09365060

Principles of Automatic Control (1)

自动控制原理1

Topic 7

Root Locus Techniques

(Chapter 8 in text book)

顾申申 Shenshen (Kevin) Gu

Ph.D. (CUHK), Associate Professor

Department of Automation

Shanghai University

gushenshen@shu.edu.cn

Review for the previous topics

- Three objectives in designing a control system (Topic 1):
  - Transient response (Topic 3);
  - Stability (Topic 5);
  - Steady state error (Topic 6).

- **What is the most important step in transient response, stability and steady state error analysis?**

Stability, steady state error, and the characteristics of transient response of closed-loop systems

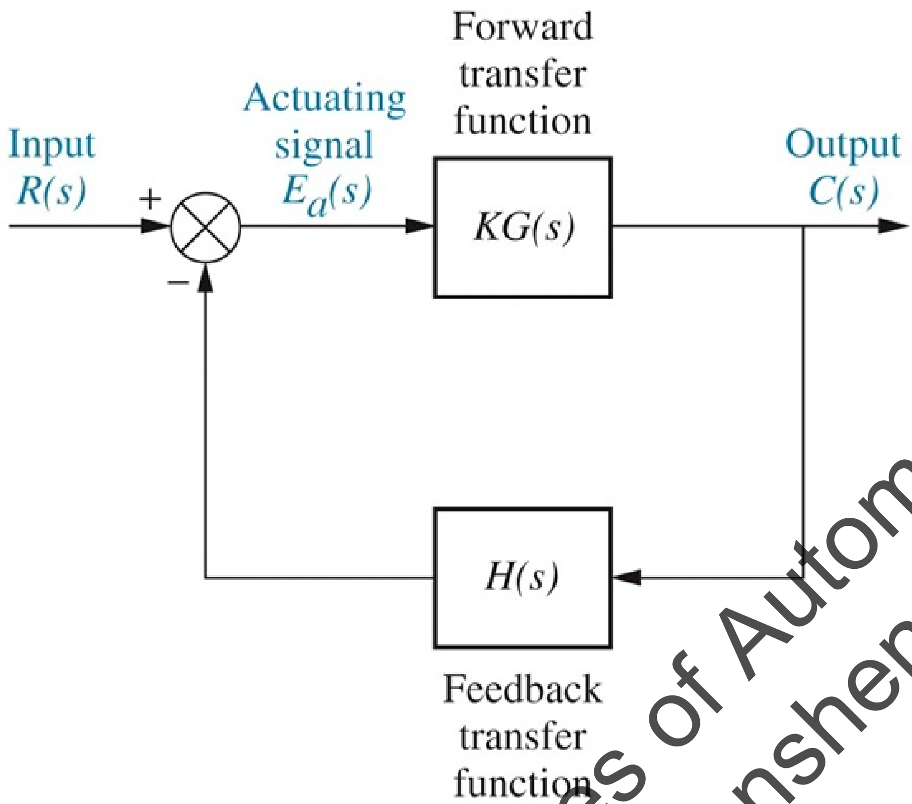
⟷

Locations of the closed-loop poles

Problems to find the location of the poles or find the roots of the denominator of TF **(characteristic equation):**

1. Difficult for a system of third or higher order. (sys. analysis)
2. Tedious for varying parameters. (sys. design)

Varying the loop gain K



The open loop gain *K* is an important parameter that can affect the performance of a system

- In many systems, simple gain adjustment may move the closed-loop poles to desired locations.
- Then the design problem may become the selection of an appropriate gain value.
- It is important to know how the closed-loop poles move in the *s* plane as the loop gain *K* is varied.

New terminologies in this topic

- Locus 轨迹
- Loci 轨迹(Plural form of locus)
- Root locus 根轨迹
- Branch 分支
- Real-axis segments 实轴部分
- Sketch 绘制
- Angle of departure 起始角
- Angle of arrival 终止角
- Asymptote 渐进线
- Breakaway point 分离点
- Starting point 起始点
- Ending point 终止点
- Characteristic equation 特征方程
- Magnitude 模
- Angle 相角
- Cartesian coordinate 笛卡尔坐标

Learning Outcomes for Topic 7

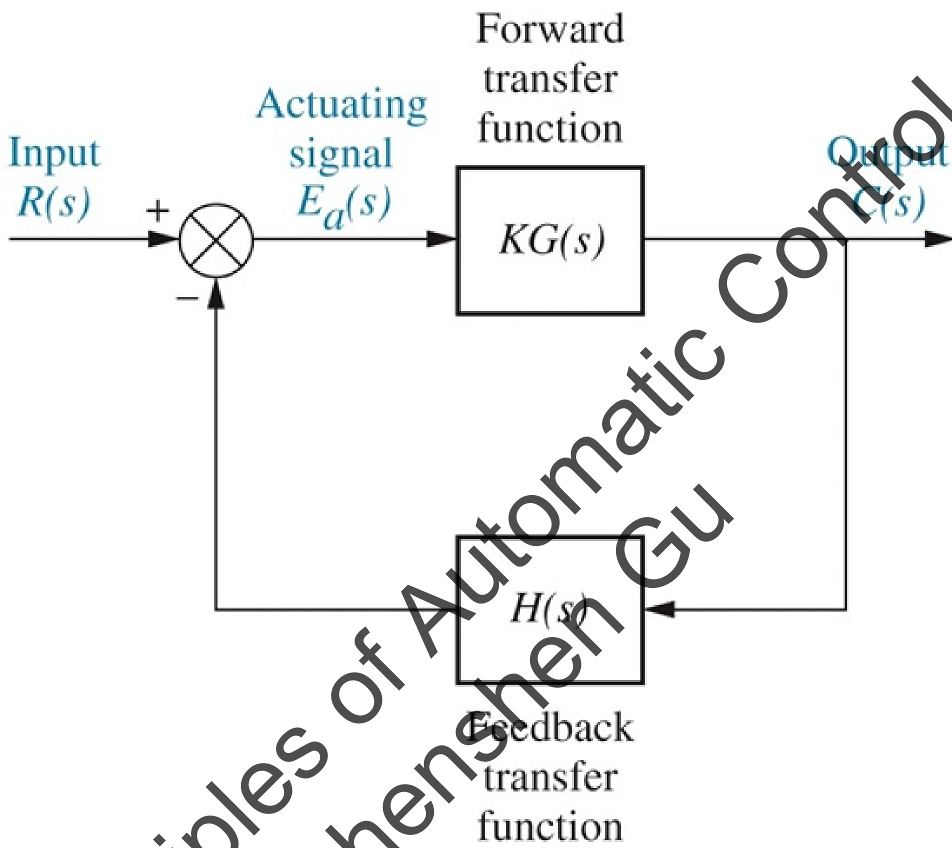
After completing this topic, you will be able to:

- Define a root locus
- State the properties of a root locus
- Sketch a root locus
- Find the coordinates of points on the root locus and their associated gains

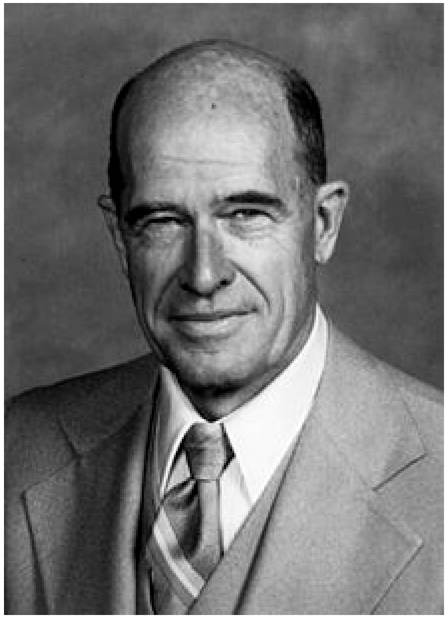
Outline

- Brief Introduction
- Defining the Root Locus
- Properties of the Root Locus
- Sketching the Root Locus
- Refining the Sketch
- An Example
- Summary

Brief Introduction



- Feedback control systems are difficult to comprehend from a qualitative point of view, and hence they rely heavily upon mathematics.



Walter Richard Evans (1920-1999)

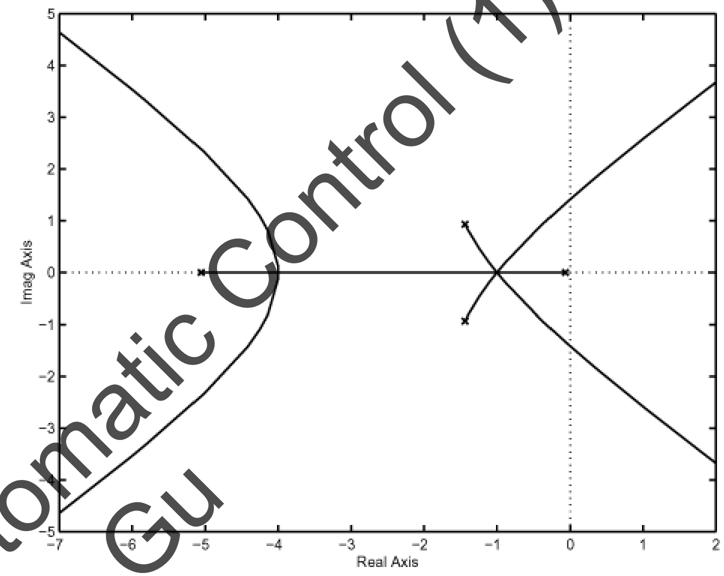
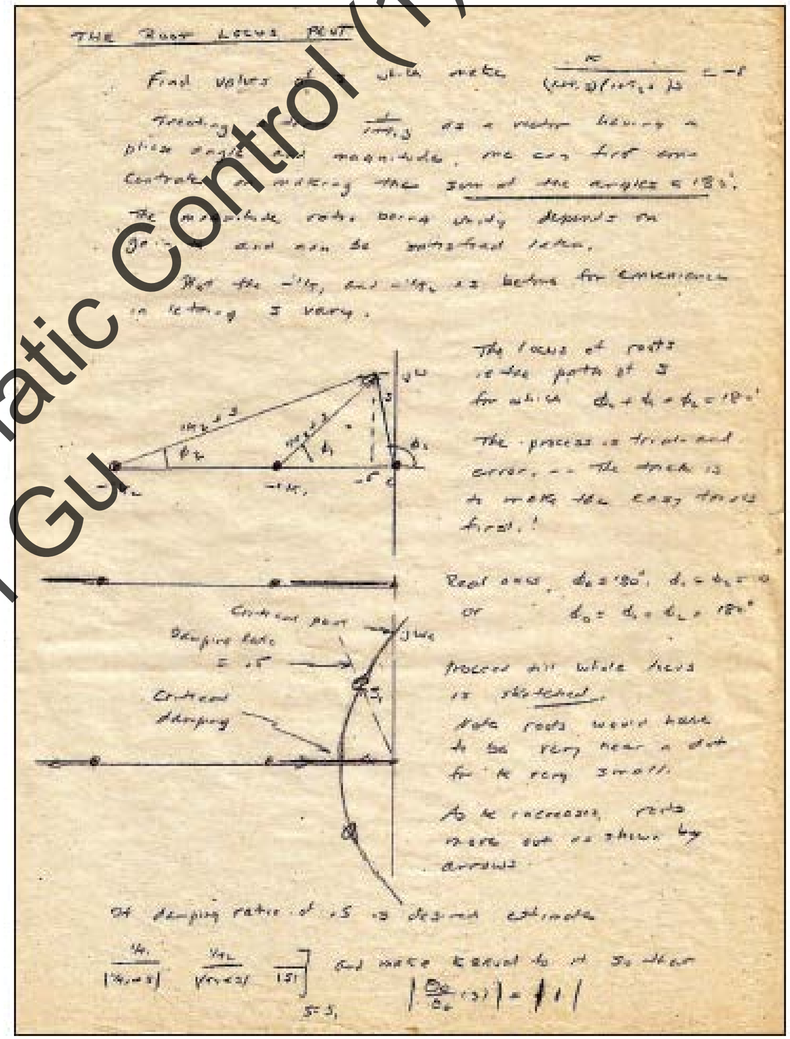


Figure 4: Root locus for $G(s) = 1/(s^4 + 8s^3 + 18s^2 + 16s + 1)$.

- From 1948 to 1950, W.R. Evans proposed a method called **ROOT LOCUS**. Root locus is a graphical presentation of the closed-loop poles as a system parameter is varied.
- Root locus is a powerful method of analysis and design for stability and transient response. It gives us the qualitative description of a control system's performance.

Walter Richard Evans and His Manuscript

Walter R. Evans as a student in the 1940s



THE ROOT LOCUS PLOT

The locus of roots is the path of s for which ... = 180°

The process is trial and error, -- the trick is to make the easy tries first!

Real axis

Process all while hers is sketched.

Note roots would have to be very near a dot for K very small.

As K increases, roots move out as shown by arrows

Two Important References

Graphical Analysis of Control Systems

WALTER R. EVANS
ASSOCIATE AIEE

Synopsis: The purpose of this paper is to demonstrate some graphical methods for finding the transient response of a control system. A simple position follow-up system is considered for convenience although the method is applicable in the same form for higher order systems or those in which only empirical frequency data is known. The basic procedure is to find the roots of the differential equation which correspond to the exponential transient terms which dominate the response. Doctor Profos[1] of Switzerland points out that the plot of the function which describes the system from error to output is a function of a complex variable of which frequency is the imaginary part and damping is the real part. The Nyquist plot is thus one line of a conformal map with the root of the equation being the value of the variable which makes the function equal to −1. Any line of plot can be calculated for systems with known functions with essentially the same ease as the Nyquist plot by use of some graphical tricks. The amplitude of any transient term is determined from the plot once the root is known by use of a theorem of operational calculus. The development possibilities of the subject seem to be very great as suggested by several topics not yet investigated.

Review of Fundamentals

A QUADRATIC SYSTEM will first be analyzed in order to emphasize the important concepts in finding any transient response. Consider the position follow-up system shown in Figure 1.

The differential equation relating the output to the error is

$$K\epsilon=\left(1+T_M\frac{d}{dt}\right)\frac{d}{dt}\theta_o(t) \quad (1)$$

K is the output speed corresponding to a unit error. T_M is the time constant of motor acceleration, other delays are neglected. But input is the output plus the error.

$$\theta_i(t)=\theta_o(t)+\epsilon(t)=\left[1+\frac{1}{K}\left(1+T_M\frac{d}{dt}\right)\frac{d}{dt}\right]\theta_o(t) \quad (2)$$

Consider the input to be a unit step and note that the steady state value of output will also be unity. Assume that the output transient can be represented by exponential terms, so that $\theta_o(t)=Ae^{st}$ is substituted into the differential equation, and the common factor Ae^{st} cancelled.

$$0=1+\frac{1}{K}(1+T_Ms)s \quad (3)$$

Note that s appears at each point where d/dt had occurred before. This equation in s is an algebraic one, and any value of s which satisfies it represents an exponential term which can exist in the transient.

Anticipating the fact that s will replace d/dt when an exponential solution is assumed, the system itself can be more conveniently represented by the block diagram of Figure 2. The function in a block represents the ratio of its output to its input. The relationship between θ_i and θ_o can now be set up directly.

$$\frac{\theta_i}{\theta_o}=\frac{\theta_o+\epsilon}{\theta_o}=1+\frac{\epsilon}{\theta_o}=1+\frac{1}{K}(1+T_Ms)s \quad (4)$$

For this case of a quadratic equation the roots can be found directly by completing the square and solving for s.

$$s=-\frac{1}{2T_M}\pm j\sqrt{\frac{K}{T_M}-\left(\frac{1}{2T_M}\right)^2}=-\sigma_n\pm j\omega_n \quad (5)$$

The oscillatory case is taken because it is typical of the response of a fast control system. The transient solution is the sum of two exponentials, one for each root.

$$A_1e^{(-\sigma_n+j\omega_n)t}+A_2e^{(-\sigma_n-j\omega_n)t} \quad (6)$$

But this can be converted to a cosine function using the relation

$$e^{j\omega t}=\cos\omega t+j\sin\omega t \quad (7)$$

Select as the new undetermined constants the amplitude A and the phase angle ϕ

$$Ae^{-\sigma_nt}\cos(\omega_nt-\phi) \quad (8)$$

The constants are determined from the initial conditions that the output is zero and its rate of change is zero at time zero. The complete solution for output then becomes:

$$\theta_o(t)=1-\left(\frac{1}{\cos\phi}\right)e^{-\sigma_nt}\cos(\omega_nt-\phi);\quad \text{in which } \tan\phi=\frac{\sigma_n}{\omega_n} \quad (9)$$

The following numerical values are selected for convenience

$$K=2/\text{seconds}\quad T_M=1\text{ second} \quad (10)$$

Note that if T_M were equal to one-tenth second, the value of θ_i/θ_o given in equation 4 would be the same if s and K were both made ten times larger. Thus, the results of these problems can be shifted into any range of values with which the reader may be normally accustomed. Substituting the foregoing values gives

$$\theta_o(t)=1-1.07e^{-0.5t}\cos(1.32t-21°) \quad (11)$$

Graphical Plot to Locate Real Roots

The consideration of an additional delay in the control system raises the degree of the equation from second to third. In this case, consider the delay to be the time constant of the inductive build-up of current in the field of the generator supplying the motor. Setting up the ratio θ_i/θ_o

$$\frac{\theta_i}{\theta_o}=1+\frac{\epsilon}{\theta_o}=1+\frac{1}{K}(1+T_Gs)(1+T_Ms)s \quad (12)$$

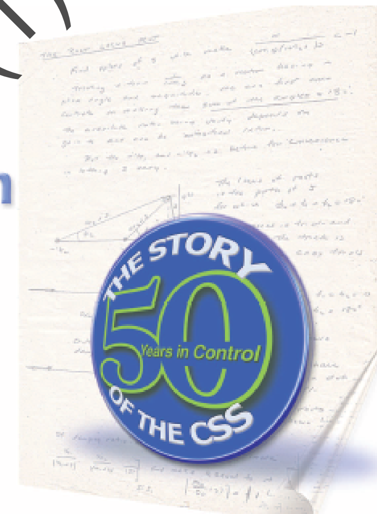
The previous values of $K=2, T_M=1$ will be kept and T_G taken as one-fourth second.

Paper 48-85, recommended by the AIEE basic sciences committee and the joint subcommittee on servomechanisms and approved by the AIEE technical program committee for presentation at the AIEE winter general meeting, Pittsburgh, Pa., January 26–30, 1948. Manuscript submitted August 11, 1947; made available for printing December 29, 1947.

WALTER R. EVANS, assistant professor, department of electrical engineering, Washington University, St. Louis, Mo.

FEATURE

Bringing Root Locus to The Classroom



The story of Walter R. Evans and his textbook *Control-System Dynamics*

by Gregory Walter Evans

Fifty years ago, on 30 September 1954, the McGraw-Hill Company published *Control-System Dynamics* by Walter R. Evans as the 40th entry in its popular "Electrical and Electronic Engineering Series." Six years earlier Evans had helped usher in a revolution in the practice of servomechanism design with his root-locus method. Engineers trained in its use were able to shape the dynamic responses of their servo designs starting from the poles and zeroes of their characteristic equations. Engineering educators would need a new textbook to describe the method to their students. With *Control-System Dynamics*, Evans sought to satisfy that need with a book emphasizing the physical picture over mathematical formalisms. Its development took him on a journey over a road less traveled.

1941–1948: Lamentations and Foundations

In 1948 John Wiley and Sons published a book destined to become a classic: *Principles of Servomechanisms* by Gordon Brown and Don Campbell of MIT. The authors acknowledged the importance of the characteristic equation's roots but lamented the obstacles confronting all known algebraic methods of solving for them, writing "Therefore, with the exception of being able to treat the most elementary forms, the whole structure of algebraic synthesis is of very small value" [1].

Brown and Campbell referred readers who were interested in learning rules for calculating the roots of characteristic equations to the book *Mathematics of Modern Engineering*. Its authors, Robert Doherty and Ernest Keller, had developed it for courses they taught at the General Electric Company's "Advanced Course," a program Doherty founded in 1922. Ironically, in their "Forward for Instructors," the book's authors lamented the practice of teaching mathematical rules rather than mathematical reasoning, writing that

> The use of mathematics as a tool in straight thinking is, in the authors' opinion, not at all what it should be. . . . Merely to remember some formula and the type of problem to which [the results of others] are applicable is of little avail in solving a new, practical problem. For the latter, not only is a knowledge of the basic engineering sciences necessary, but also a mind disciplined in sound reasoning. [2]

Having defined the problem, the authors offered their recommended solution:

> Often the clearest approach to a mathematical concept, function, or theory is through the solution of an introductory problem, rather than through a formal approach by means of definitions, axioms, and theorems. . . . An engineer is interested primarily in the application of mathematics to the solution of problems. [2]

Evans was a product of Robert Doherty's philosophy, both taking and teaching classes in the Advanced Course from 1941 to 1946. (See Figure 1.) There he mastered methods for attacking problems using principles rather than

0272-1708/04/$20.00©2004IEEE

Why we need Root Locus ?

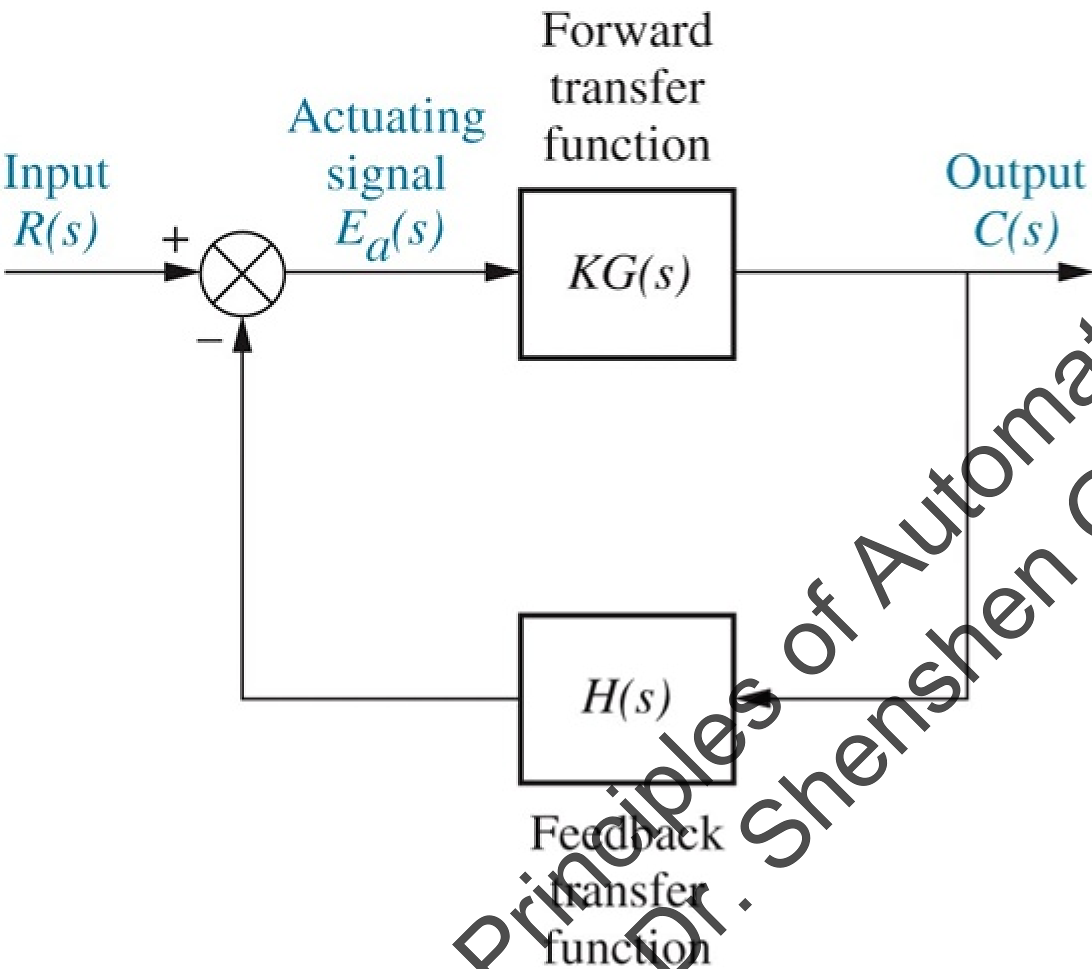
- If the open loop transfer function does not provide the response we want and there are disturbances, then a feedback system is required. Invariably a controller (compensator) need to be designed to satisfy the specification requirements.
- Suppose we wanted to include a simple gain amplifier, we need to test several values of the gain to see which one is appropriate.
- However, if we can get a picture of how the roots of the characteristic equation will behave when we change the gain value, it would be easier to select a suitable gain value.
- This can be obtained by sketching the root locus. Evans found a list of rules that can be used to obtain the sketch.

What will Root Locus do?

- Root locus is a graphical method of tracing the path of the poles as you change the system gain in a feedback system.
- Predicts effect of closing the loop.
- Predicts effect of changing gain.
- Predicts effect of adding more poles and zeroes.

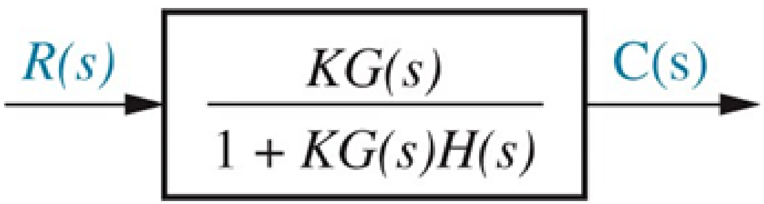
- Before presenting root locus, let us review two concepts that we need for the ensuing discussion:
  - The Control System Problem
  - Vector Presentation of Complex Numbers

The Control System Problem



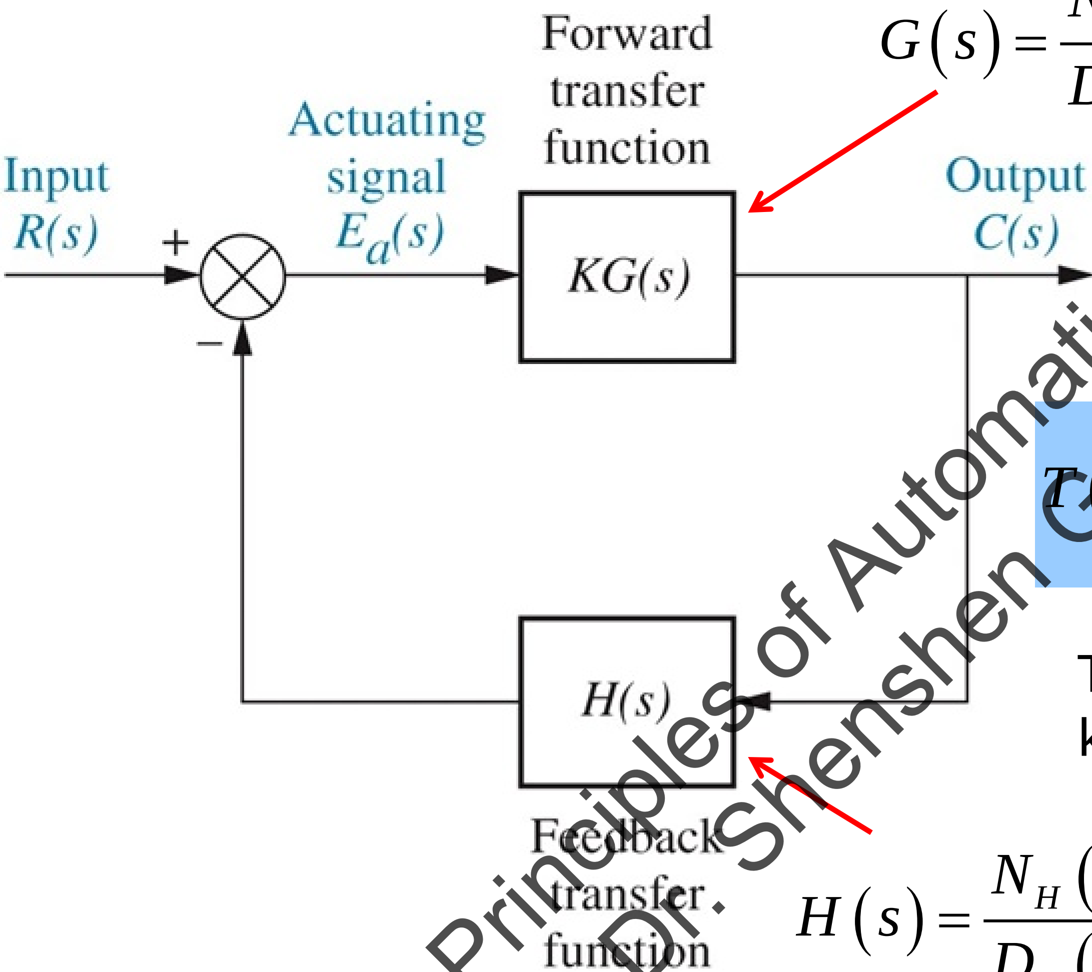
Open-loop transfer function: *KG(s)H(s)*

We can determine the poles of *KG(s)H(s)* conveniently. Poles do not change when *K* changes.



We cannot determine the poles directly unless we factor the denominator. Also, the poles change with *K*.

$$G(s)=\frac{N_G(s)}{D_G(s)}$$



Zeros of *T(s)* consist of the zeros of G(s) and the poles of *H(s)*.

$$T(s)=\frac{KN_G(s)D_H(s)}{D_G(s)D_H(s)+KN_G(s)N_H(s)}$$

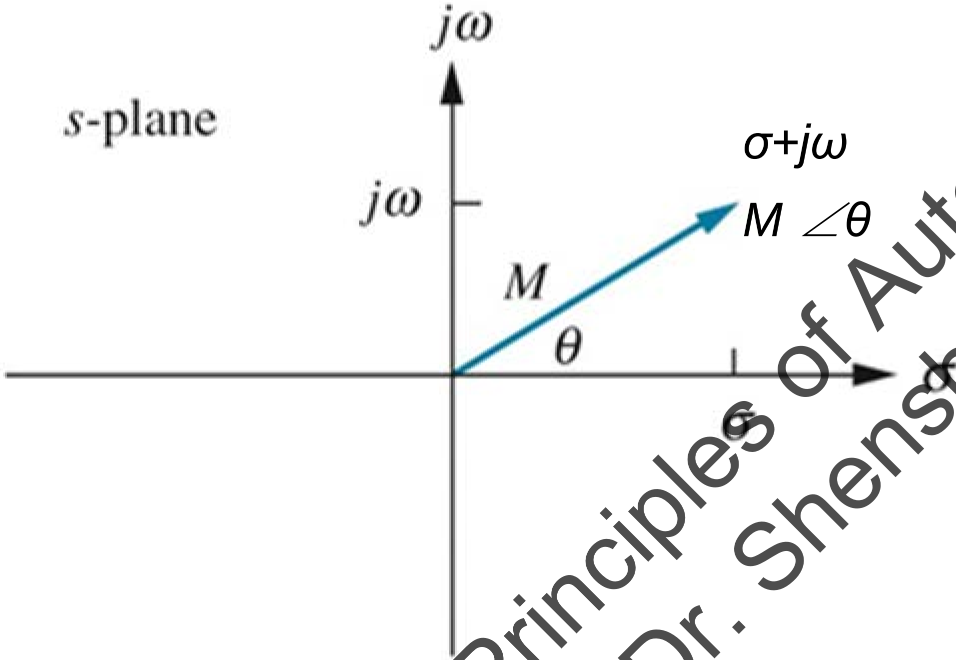
The poles of *T(s)* are not immediately known and in fact can change with *K*.

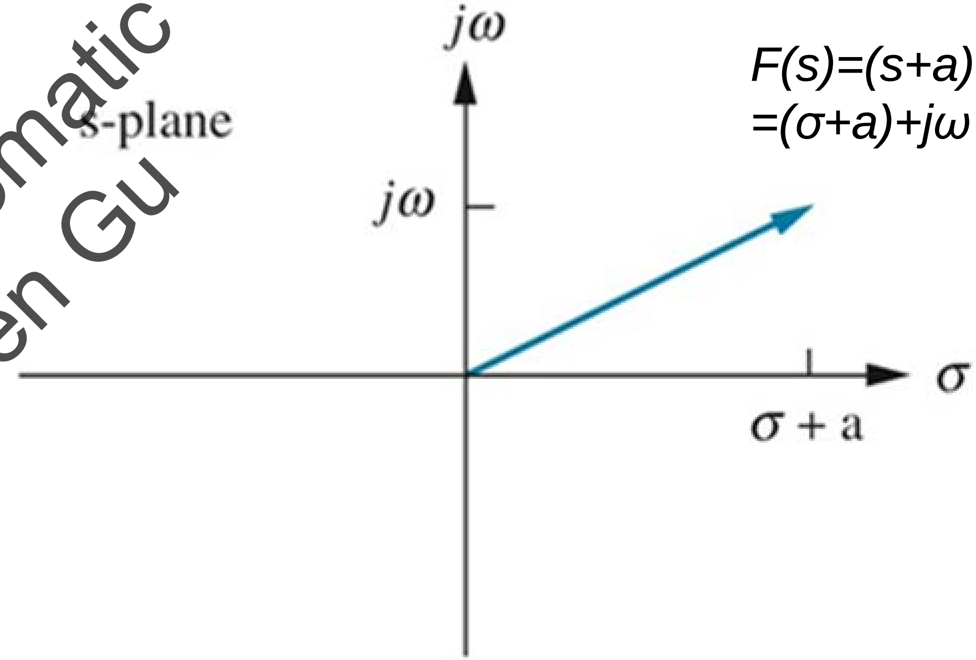
$$H(s)=\frac{N_H(s)}{D_H(s)}$$

Vector Presentation of Complex Numbers

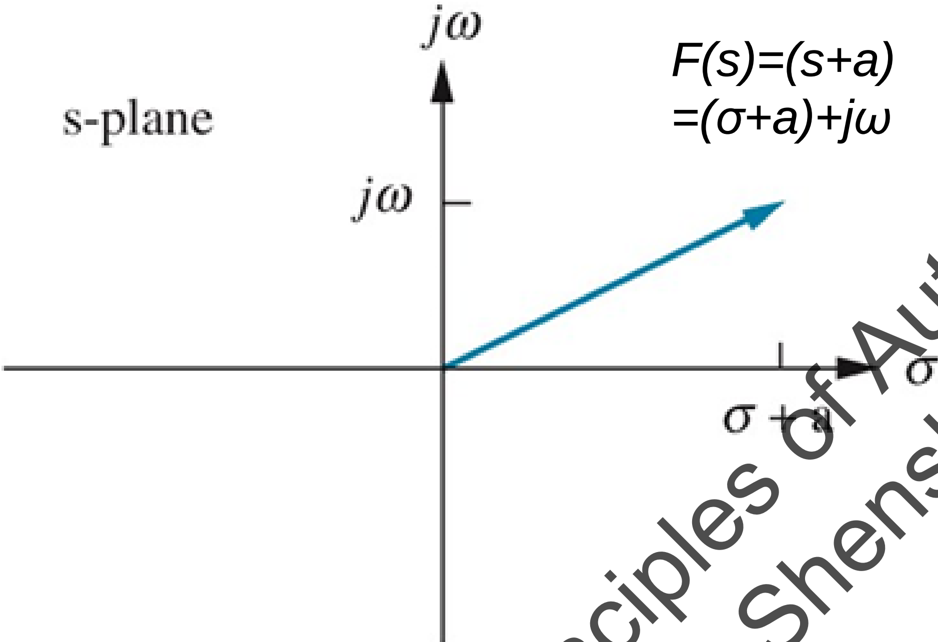
Any complex number, $\sigma + j\omega$, described in Cartesian coordinates can be graphically represented by a vector

Complex function, $F(s)$: Generate another complex number

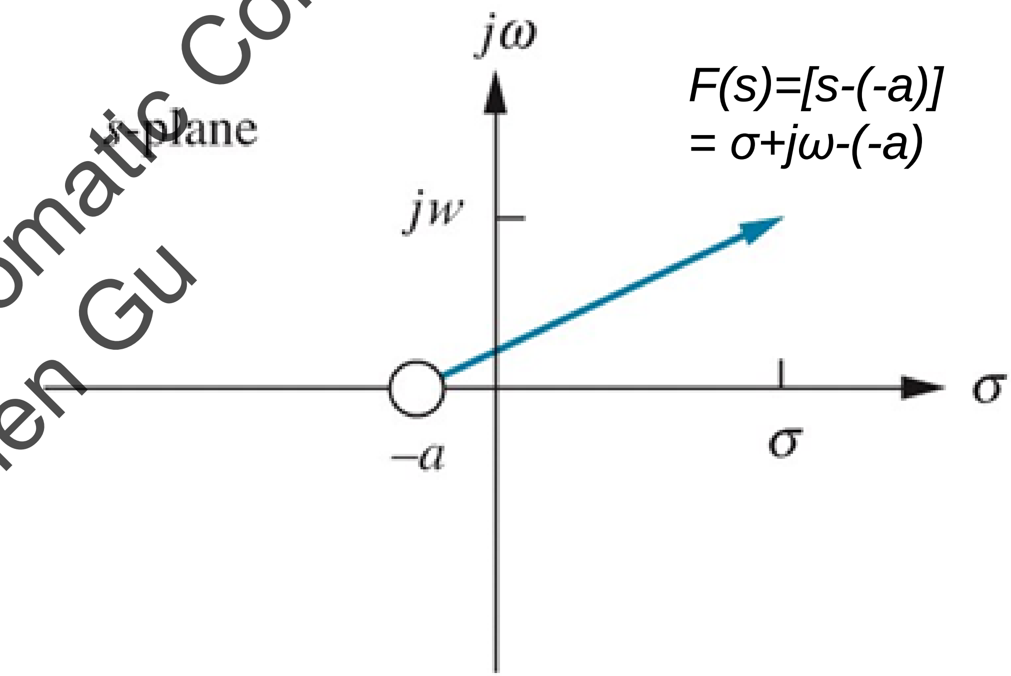




$F(s)=(s+a)$
$=(\sigma+a)+j\omega$

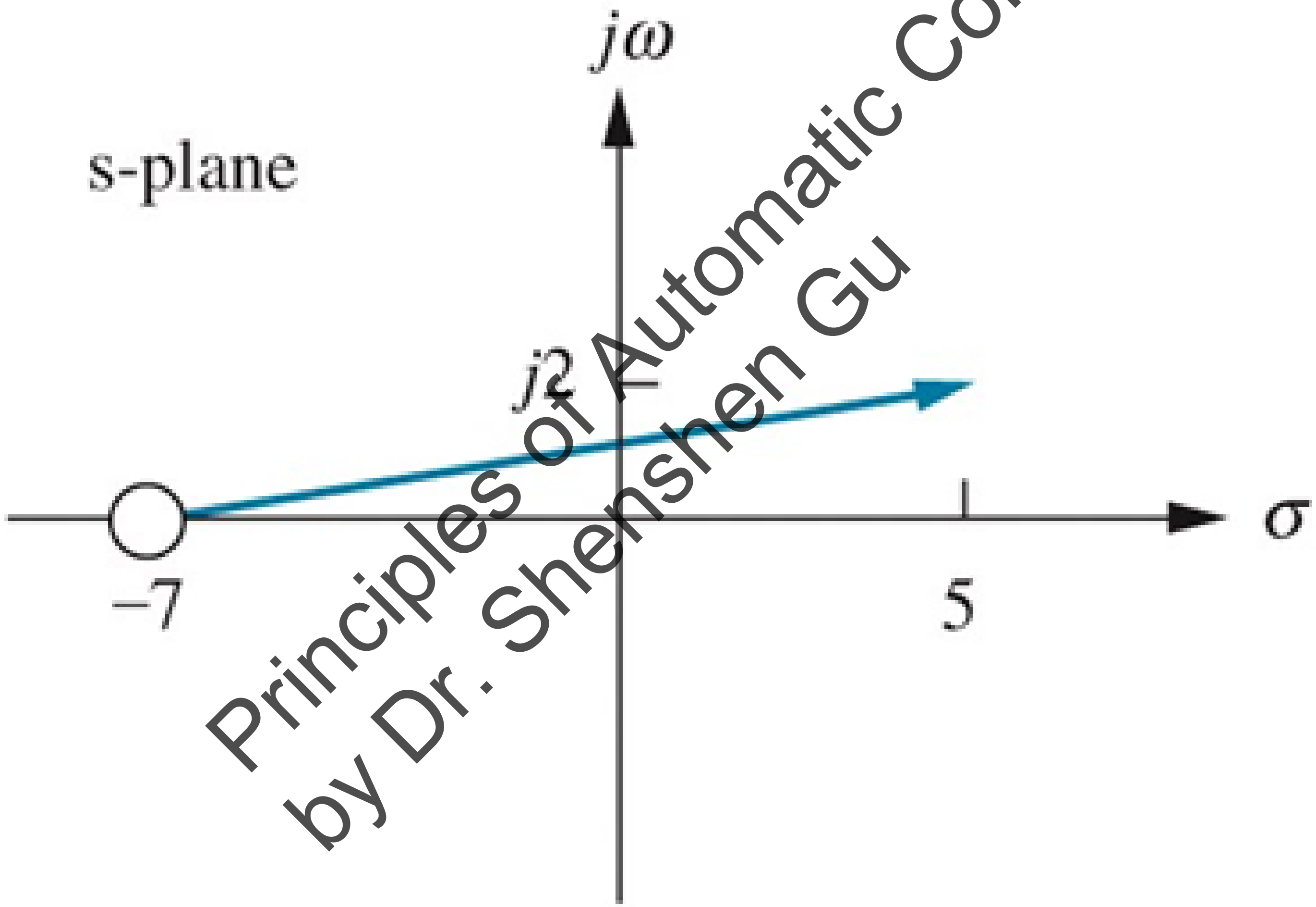


$F(s)=[s-(-a)]$
$= \sigma+j\omega-(-a)$



Quiz

$$(s+7)\Big|_{s\to 5+j2}$$



$$F(s)=\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)}=\frac{\prod \text{numerator's complex factors}}{\prod \text{denominator's complex factors}}$$

The Symbol $\prod$ means "product," m=number of zeros; and n=number of poles.

$$M=\frac{\prod \text{zero lengths}}{\prod \text{pole lengths}}=\frac{\prod_{i=1}^{m}\left|(s+z_i)\right|}{\prod_{j=1}^{n}\left|(s+p_j)\right|}$$

Zero length, $|(s+z_i)|$, is the magnitude of the vector drawn from the zero of F(s) at $-z_i$ to the point s. **Pole length**, $|(s+p_j)|$, is the magnitude of the vector drawn from the pole of F(s)at $-p_j$ to the point s.

$$\theta=\sum \text{zero angles}-\sum \text{pole angles}$$

$$=\sum_{i=1}^{m}\angle(s+z_i)-\sum_{j=1}^{n}\angle(s+p_j)$$

Zero angle is the angle, measured from the positive extension of the real axis, of a vector drawn from the zero of F(s) at $-z_i$ to the point s. **Pole angle** is the angle, measured from the positive extension of the real axis, of the vector drawn from the pole of F(s) at $-p_j$ to the point s.

Example 8.1

Evaluation of a Complex Function via Vectors

PROBLEM: Given

$$F(s) = \frac{(s+1)}{s(s+2)} \tag{8.7}$$

find $F(s)$ at the point $s = -3 + j4$.

SOLUTION: The problem is graphically depicted in Figure 8.3, where each vector, $(s+\alpha)$, of the function is shown terminating on the selected point $s = -3 + j4$. The vector originating at the zero at -1 is

$$\sqrt{20}\angle 116.6^\circ \tag{8.8}$$

The vector originating at the pole at the origin is

$$5\angle 126.9^\circ \tag{8.9}$$

The vector originating at the pole at -2 is

$$\sqrt{17}\angle 104.0^\circ \tag{8.10}$$

Substituting Eqs. (8.8) through (8.10) into Eqs. (8.5) and (8.6) yields

$$M\angle\theta = \frac{\sqrt{20}}{5\sqrt{17}}\angle 116.6^\circ - 126.9^\circ - 104.0^\circ = 0.217\angle -114.3^\circ \tag{8.11}$$

as the result for evaluating $F(s)$ at the point $-3 + j4$.

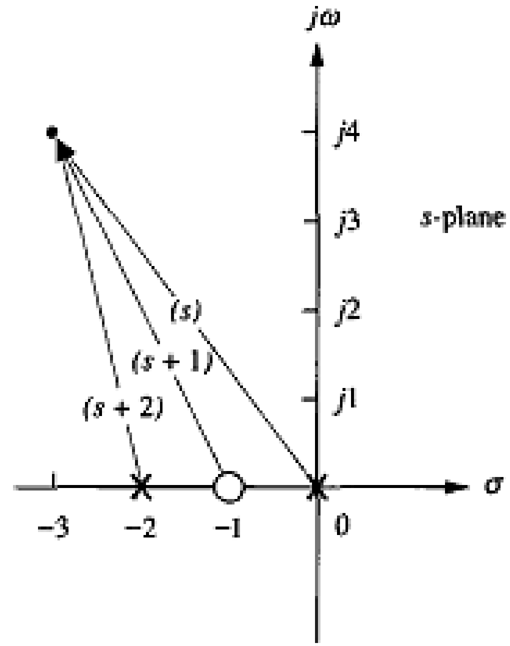
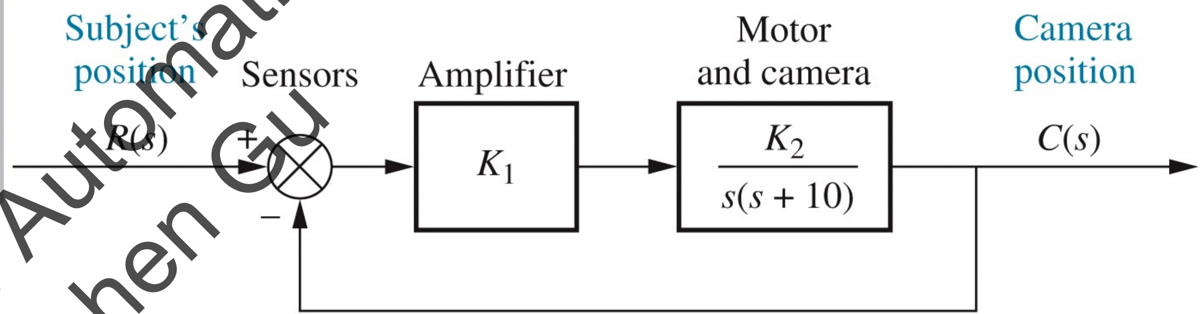


FIGURE 8.3 Vector representation of Eq. (8.7)

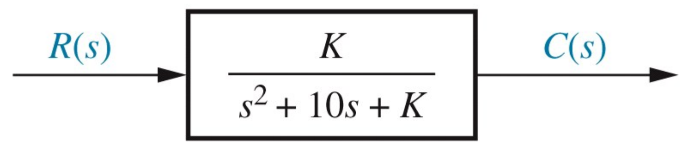
Defining the Root Locus

(*a*)

Security Camera System



(*b*)



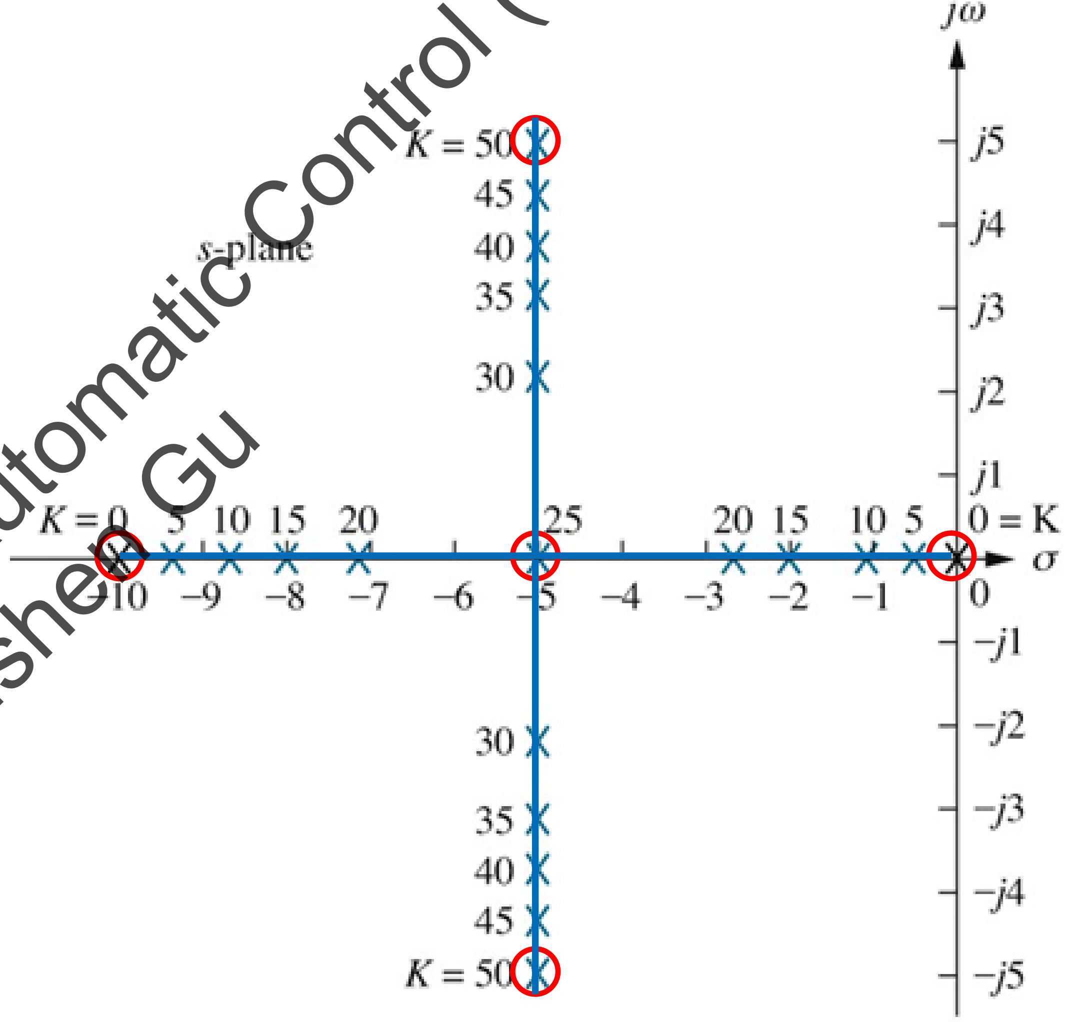
where $K = K_1K_2$

(*c*)

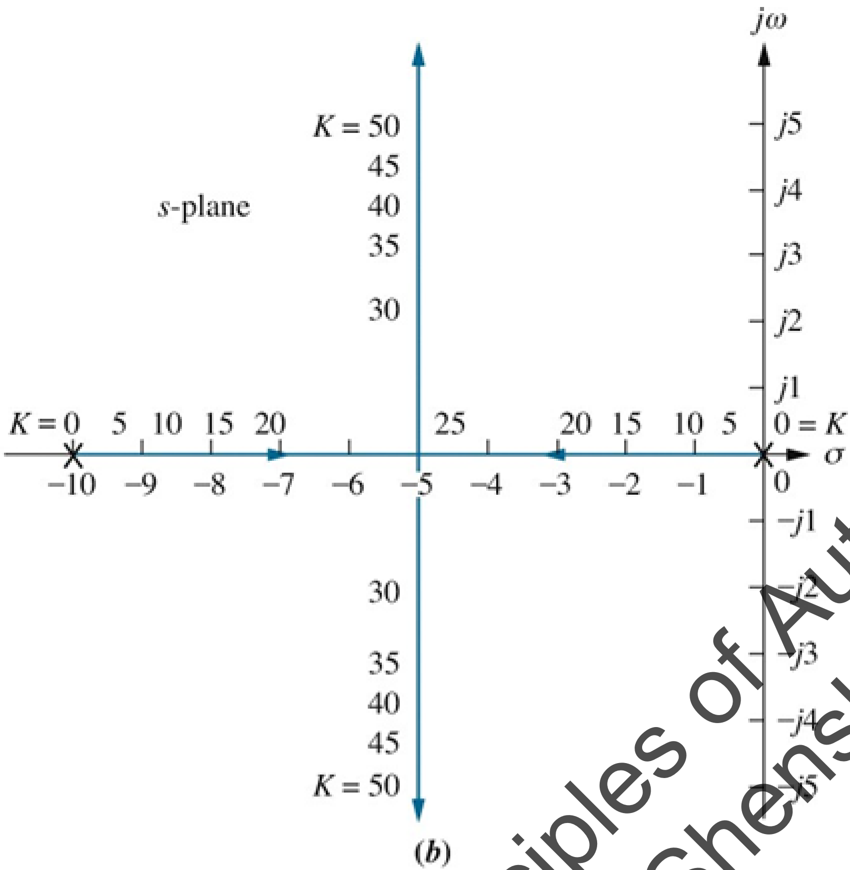
TABLE 8.1 Pole location as function of gain for the system of Figure 8.4

K	Pole 1	Pole 2
0	-10	0
5	-9.47	-0.53
10	-8.87	-1.13
15	-8.16	-1.84
20	-7.24	-2.76
25	-5	-5
30	$-5+j2.24$	$-5-j2.24$
35	$-5+j3.16$	$-5-j3.16$
40	$-5+j3.87$	$-5-j3.87$
45	$-5+j4.47$	$-5-j4.47$
50	$-5+j5$	$-5-j5$

Table 8.1



(a)



(b)

It is this representation of the paths of the closed-loop poles as the gain is varied that we call a **root locus**.

$$K \geq 0$$

- The root locus shows the changes in the transient response as the gain, K, varies
  - K<25: Overdamped
  - K=25: Critically damped
  - K>25: Underdamped
- Other Conclusions
  - K>25, underdamped: the real parts are always the same -> settling time remains the same;
  - K increases -> damping ration diminishes -> percent overshoot increases
  - K increases -> damped frequency increases ->reduction of the peak time
  - Root locus never crosses over in the right half-plane -> the system is always stable.

Properties of the Root Locus

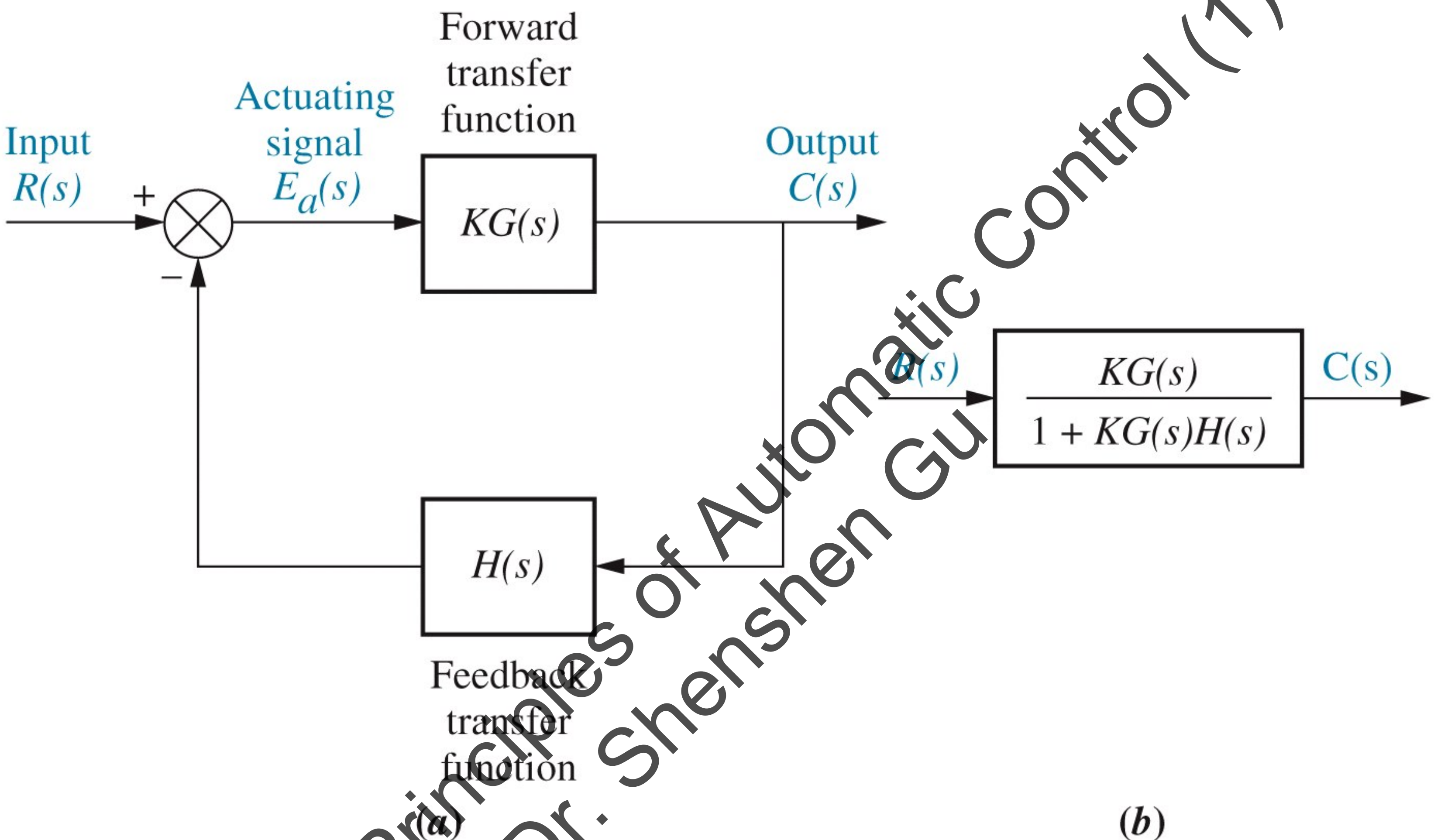


Figure 8.1

$$T(s) = \frac{KG(s)}{1 + KG(s)H(s)}$$

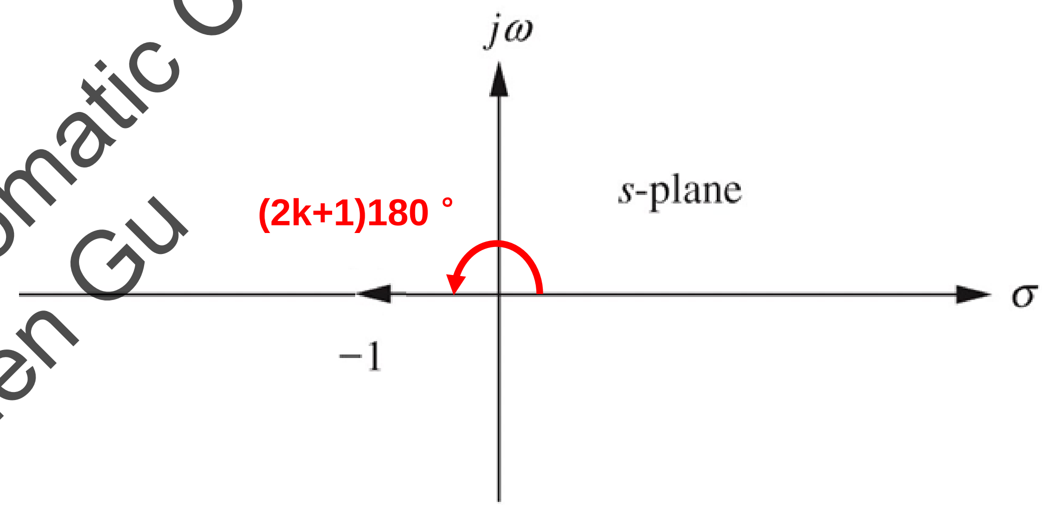
Poles of the closed-loop system

$$1 + KG(s)H(s) = 0$$

$$KG(s)H(s) = -1$$

$$KG(s)H(s) = -1 = 1\angle(2k+1)180^\circ$$

$$\begin{cases} |KG(s)H(s)| = 1 \\ \angle KG(s)H(s) = (2k+1)180^\circ \end{cases}$$

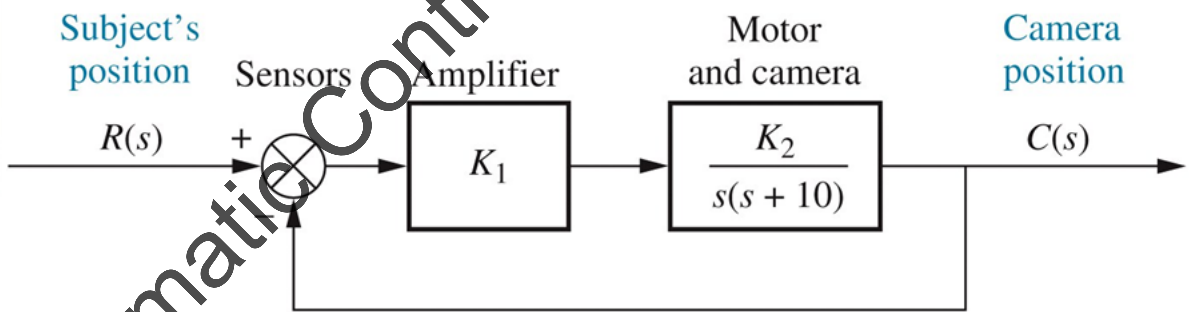


- If a value of *s* is substituted into the function *KG(s)H(s)*, a complex number results.
- If the angle of the complex number is an odd multiple of *180°*, that value of *s* is a system pole for some particular value of *K*.
- What value of *K*?

$$K = \frac{1}{\left|G(s)\right|\left|H(s)\right|}$$

TABLE 8.1 Pole location as function of gain for the system of Figure 8.4

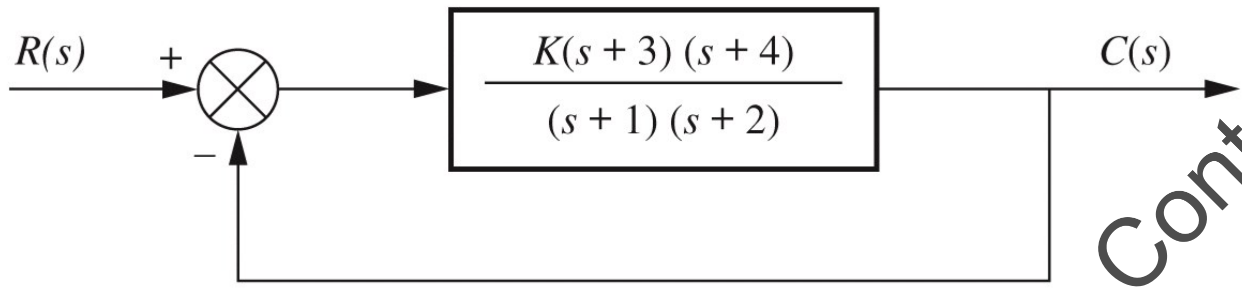
K	Pole 1	Pole 2
0	−10	0
5	−9.47	−0.53
10	−8.87	−1.13
15	−8.16	−1.84
20	−7.24	−2.76
25	−5	−5
30	$-5 + j2.24$	$-5 - j2.24$
35	$-5 + j3.16$	$-5 - j3.16$
40	$-5 + j3.87$	$-5 - j3.87$
45	$-5 + j4.47$	$-5 - j4.47$
50	$-5 + j5$	$-5 - j5$



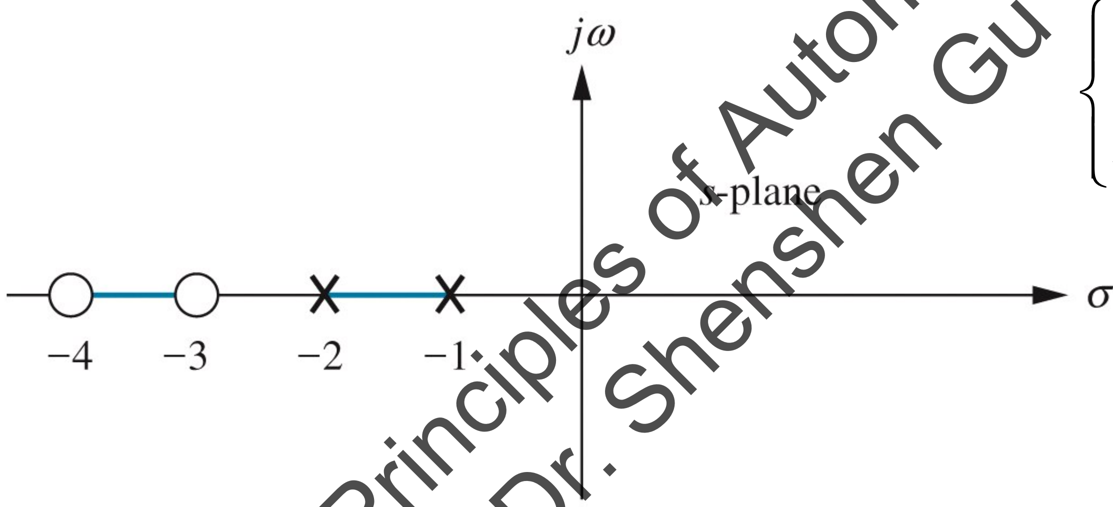
$$KG(s)H(s) = \frac{K}{s(s+10)}$$

$$KG(s)H(s)\Big|_{K=5, s=-9.47}$$

$$= \frac{5}{-9.47(-9.47+10)} = -1$$



(*a*)



(*b*)

$$
\begin{cases} \left|KG(s)H(s)\right| = 1 \\ \angle KG(s)H(s) = (2k+1)180^\circ \end{cases}
$$

Figure 8.6

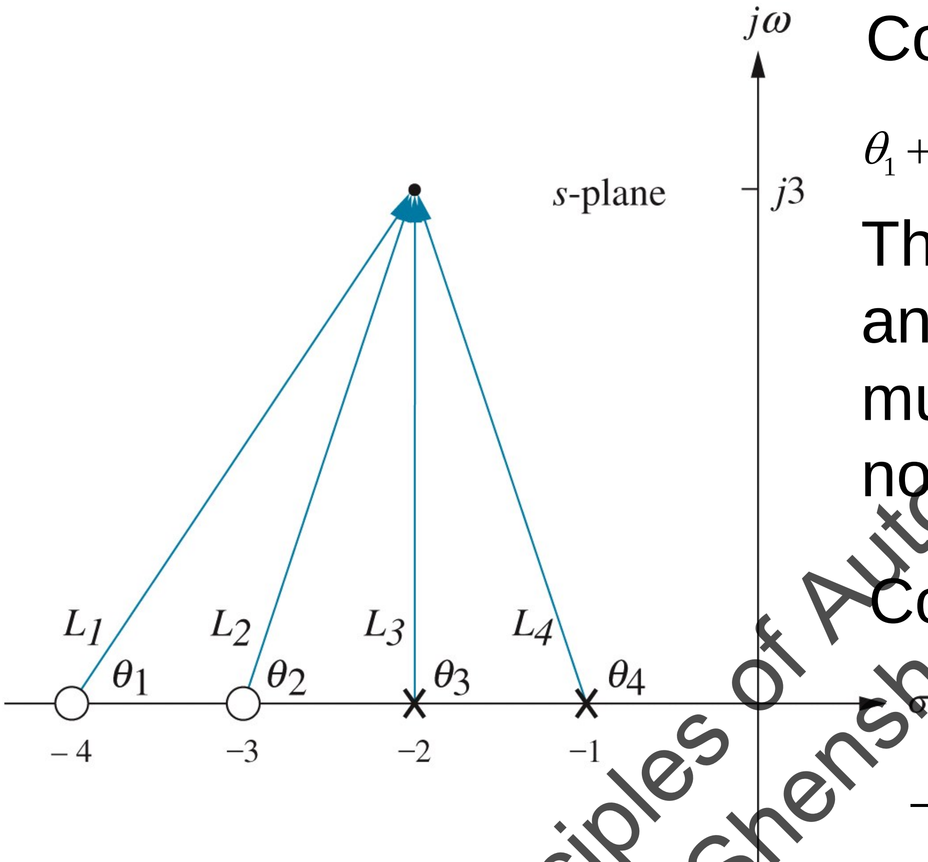


Figure 8.7

Consider the point -2+j3.

$$\theta_1+\theta_2-\theta_3-\theta_4=56.31^\circ+71.57^\circ-90^\circ-108.43^\circ=-70.55^\circ$$

The angles of the zeros minus the angles of the poles is not equal an odd multiple of 180 ° . Therefore, -2+3j is not a point on the root locus.

Consider the point $-2+j\left(\sqrt{2}/2\right)$

$$\theta_1+\theta_2-\theta_3-\theta_4=180^\circ$$

$-2+j\left(\sqrt{2}/2\right)$ is a point on the root locus for some value of gain.

$$K=\frac{L_3L_4}{L_1L_2}=\frac{\frac{\sqrt{2}}{2}(1.22)}{(2.12)(1.22)}=0.33$$

- We summarize what we have found as follows:
- Given the poles and zeros of the **open-loop transfer** function, *KG(s)H(s)*, a point in the *s*-plane is on the root locus for a particular value of gain, *K*, if the angles of the zeros minus the angles of the poles, all drawn to the selected point on the *s*-plane, add up to $(2k+1)180°$.
- Furthermore, gain K at that point for which the angles add up to $(2k+1)180°$ is found by dividing the product of the pole lengths by the product of the zero lengths.

Skill-Assessment Exercise 8.2

WileyPLUS
WPCS
Control Solutions

PROBLEM: Given a unity feedback system that has the forward transfer function

$$G(s) = \frac{K(s+2)}{(s^2+4s+13)}$$

do the following:

a. Calculate the angle of $G(s)$ at the point $(-3+j0)$ by finding the algebraic sum of angles of the vectors drawn from the zeros and poles of $G(s)$ to the given point.

b. Determine if the point specified in **a** is on the root locus.

c. If the point specified in **a** is on the root locus, find the gain, K, using the lengths of the vectors.

ANSWERS:

a. Sum of angles $= 180°$

b. Point is on the root locus

c. $K = 10$

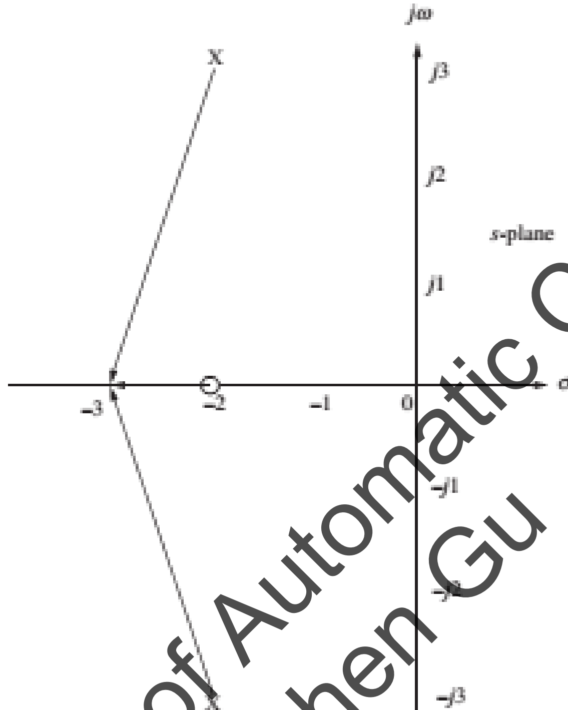
The complete solution is at www.wiley.com/college/nise.

TryIt 8.2

Use MATLAB and the following statements to solve Skill-Assessment Exercise 8.2.

```
s=-3+0j;
G=(s+2)/(s^2+4*s+13);
Theta=(180/pi)*...
 angle(G)
M=abs(G);
K=1/M
```

a. First draw the vectors.



From the diagram:

$$\sum \text{angles} = 180° - \tan^{-1}\left(\frac{-3}{-1}\right) - \tan^{-1}\left(\frac{-3}{1}\right) = 180° - 108.43° + 108.43° = 180°.$$

b. Since the angle is 180°, the point is on the root locus.

c. $K = \dfrac{\Pi\ \text{pole lengths}}{\Pi\ \text{zero lengths}} = \dfrac{\left(\sqrt{1^2+3^2}\right)\left(\sqrt{1^2+3^2}\right)}{1} = 10$

Sketching the Root Locus

- Let us study five rules that allow us to sketch the root locus using minimal calculations:
  - Number of branches
  - Symmetry
  - Real-axis segments
  - Starting and ending points
  - Behavior at infinity

Rule 1: Number of branches

- If we define a branch as the path that one pole traverses, then there will be one branch for each closed-loop pole.
- ***The number of branches of the root locus equals the number of closed-loop poles.***

Rule 2: Symmetry

- If complex closed-loop poles do not exist in conjugate pairs, the resulting polynomial, formed by multiplying the factors containing the closed-loop poles, would have complex coefficients. Physically realizable systems cannot have complex coefficients in their transfer functions. Thus, we conclude:
- ***The root locus is symmetrical about the real axis.***

Rule 3: Real-axi

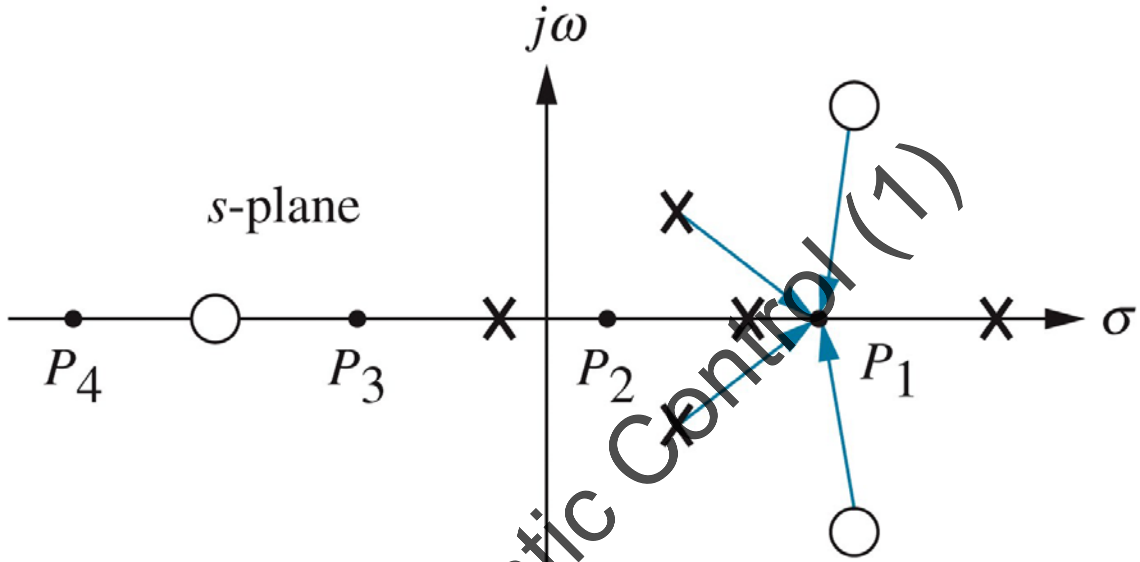


Figure 8.8

- An attempt is made to calculate the angular contribution of the poles and zeros at each point, P1, P2, P3, and P4, along the real axis, we observe the following:
  - At each point the angular contribution of a pair of open-loop complex poles or zeros is zero;
  - The contribution of the open-loop poles and open-loop zeros to the left of the respective point is zero.

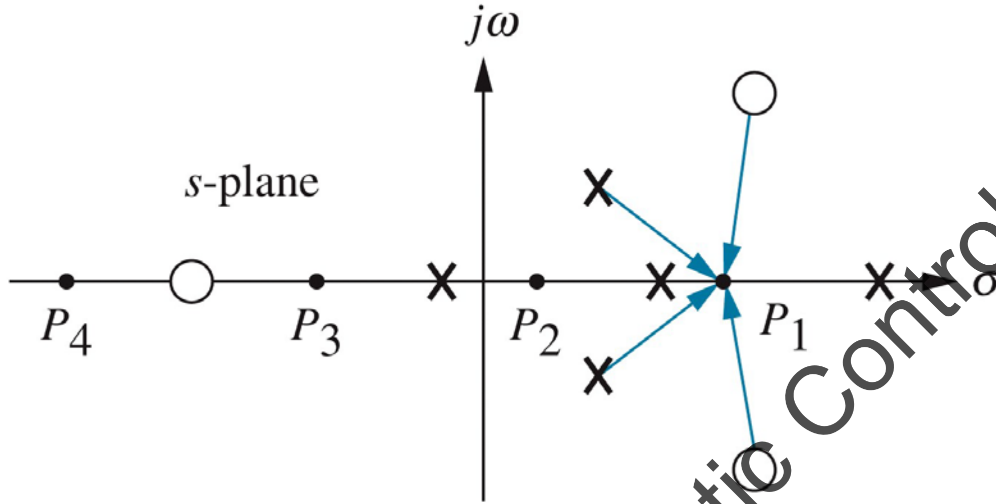


Figure 8.8

- If we calculate the angle at each point using only the open-loop, real-axis poles and zeros to the right of each point, we note the following:
  - (1) The angles on the real axis alternate between 0° and 180°, and
  - (2) the angle is 180° for regions of the real axis that exist to the left of an odd number of poles and/or zeros.
- ***On the real axis, for K > 0 the root locus exists to the left of an odd number of real axis, finite open-loop poles and/or finite open-loop zeros.***

Rule 4: Starting and ending points

Magnitude Equation: $\left|KG(s)H(s)\right| = 1$

$$\frac{K\prod_{i=1}^{m}\left|s-Z_i\right|}{\prod_{i=1}^{n}\left|s-P_i\right|} = 1 \quad \Rightarrow \quad K = \frac{\prod_{i=1}^{n}\left|s-P_i\right|}{\prod_{i=1}^{m}\left|s-Z_i\right|}$$

$K = 0 \quad \rightarrow s = P_i$ →Root loci start from poles of G(s)H(s)

$K = \infty \quad \rightarrow s = Z_i$ →Root loci end at zeros of G(s)H(s).

n>m? **Including n-m implicit zeros at infinity**

- ***The root locus begins at the finite and infinite poles of G(s)H(s) and ends at the finite and infinite zeros of G(s)H(s).***

Rule 5: Behavior at infinity

- ***The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equation of the asymptotes is given by the real-axis intercept, σ_a and angle, θ_a as follows:***

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}}$$

$$\theta_a = \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}}$$

- where $k = 0, 1, 2, \ldots, n-m-1$ and the angle is given in radians with respect to the positive extension of the real axis.

Example 8.2

Sketching a Root Locus with Asymptotes

PROBLEM: Sketch the root locus for the system shown in Figure 8.11.

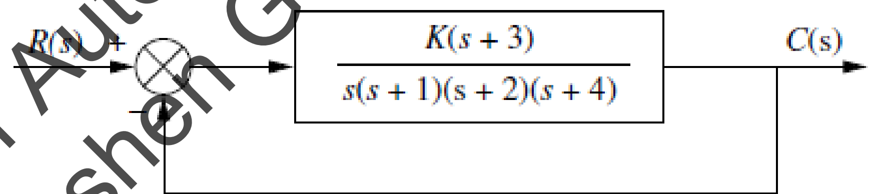


FIGURE 8.11 System for Example 8.2.

SOLUTION: Let us begin by calculating the asymptotes. Using Eq. (8.27), the real-axis intercept is evaluated as

$$\sigma_a = \frac{(-1-2-4)-(-3)}{4-1} = -\frac{4}{3} \tag{8.29}$$

The angles of the lines that intersect at $-4/3$, given by Eq. (8.28), are

$$\theta_a = \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}} \tag{8.30a}$$

$$= \pi/3 \qquad \text{for } k = 0 \tag{8.30b}$$

$$= \pi \qquad \text{for } k = 1 \tag{8.30c}$$

$$= 5\pi/3 \qquad \text{for } k = 2 \tag{8.30d}$$

If the value for k continued to increase, the angles would begin to repeat. The number of lines obtained equals the difference between the number of finite poles and the number of finite zeros.

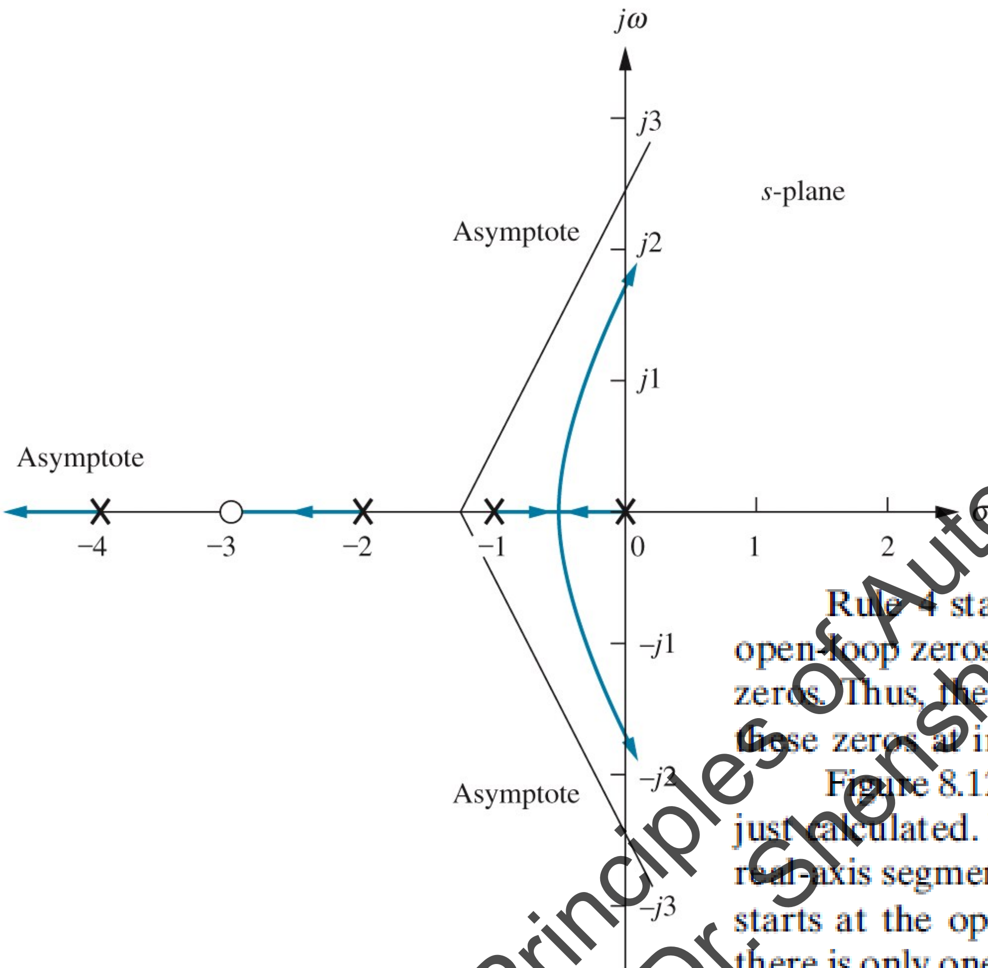


Figure 8.12

Rule 4 states that the locus begins at the open-loop poles and ends at the open-loop zeros. For the example there are more open-loop poles than open-loop zeros. Thus, there must be zeros at infinity. The asymptotes tell us how we get to these zeros at infinity.

Figure 8.12 shows the complete root locus as well as the asymptotes that were just calculated. Notice that we have made use of all the rules learned so far. The real-axis segments lie to the left of an odd number of poles and/or zeros. The locus starts at the open-loop poles and ends at the open-loop zeros. For the example there is only one open-loop finite zero and three infinite zeros. Rule 5, then, tells us that the three zeros at infinity are at the ends of the asymptotes.

Skill-Assessment Exercise 8.3

PROBLEM: Sketch the root locus and its asymptotes for a unity feedback system that has the forward transfer function

$$G(s) = \frac{K}{(s+2)(s+4)(s+6)}$$

ANSWER: The complete solution is at www.wiley.com/college/nise.

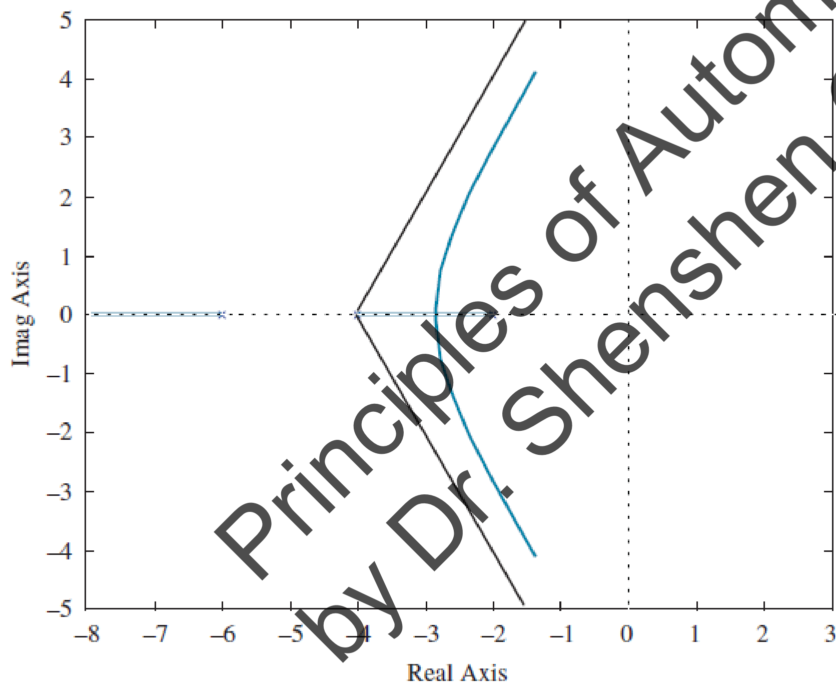
8.3

First, find the asymptotes.

$$\sigma_a = \frac{\sum \text{poles} - \sum \text{zeros}}{\#\,\text{poles} - \#\,\text{zeros}} = \frac{(-2-4-6)-(0)}{3-0} = -4$$

$$\theta_a = \frac{(2k+1)\pi}{3} = \frac{\pi}{3},\ \pi,\ \frac{5\pi}{3}$$

Next draw root locus following the rules for sketching.



Refining the Sketch

- The rules covered in the previous section permit us to sketch a root locus rapidly. If we want more detail, we must be able to accurately find important points on the root locus along with their associated gain.
  - Real-Axis Breakaway and Break-In Points
  - The $j\omega$-Axis Crossings
  - Angles of Departure and Arrival
  - Plotting and Calibrating the Root Locus

Real-Axis Breakaway and Break-In Points

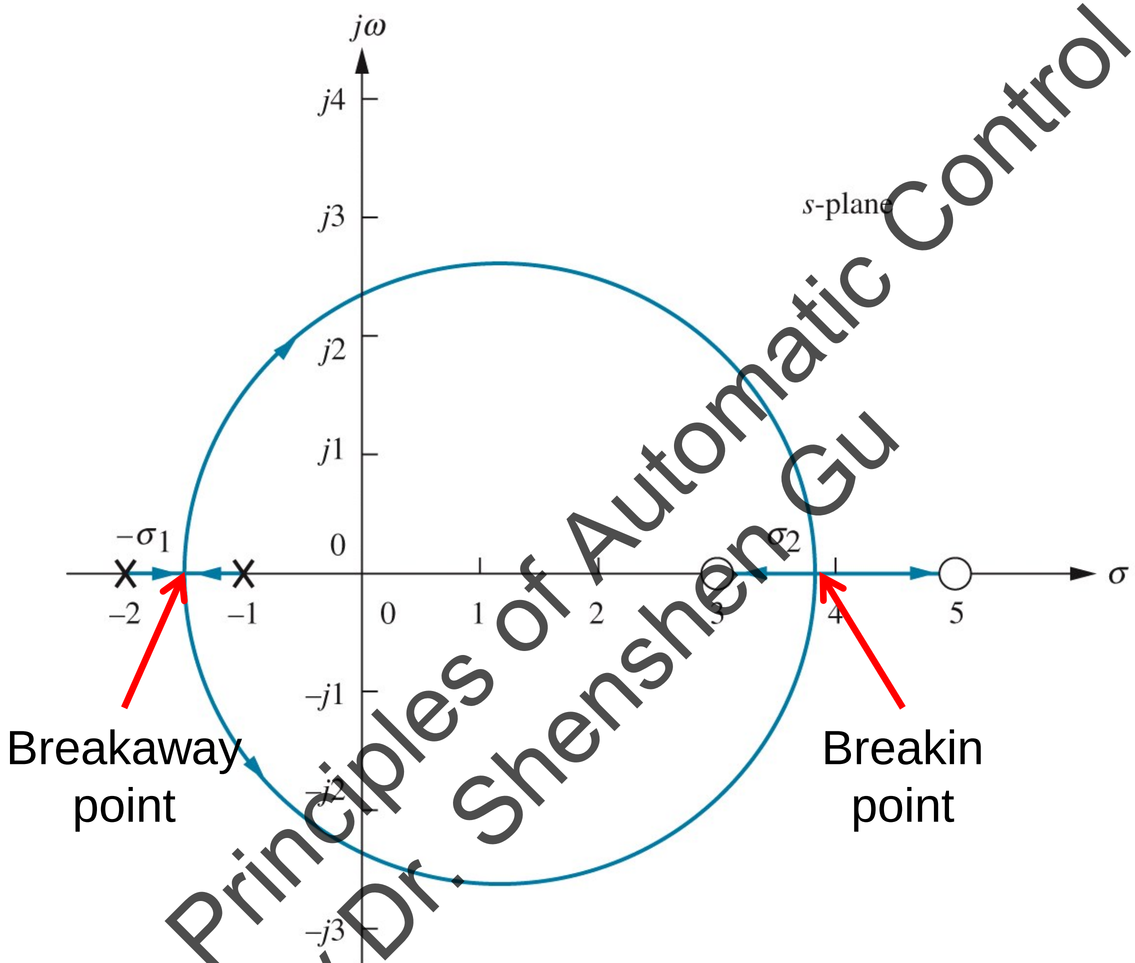


Figure 8.13

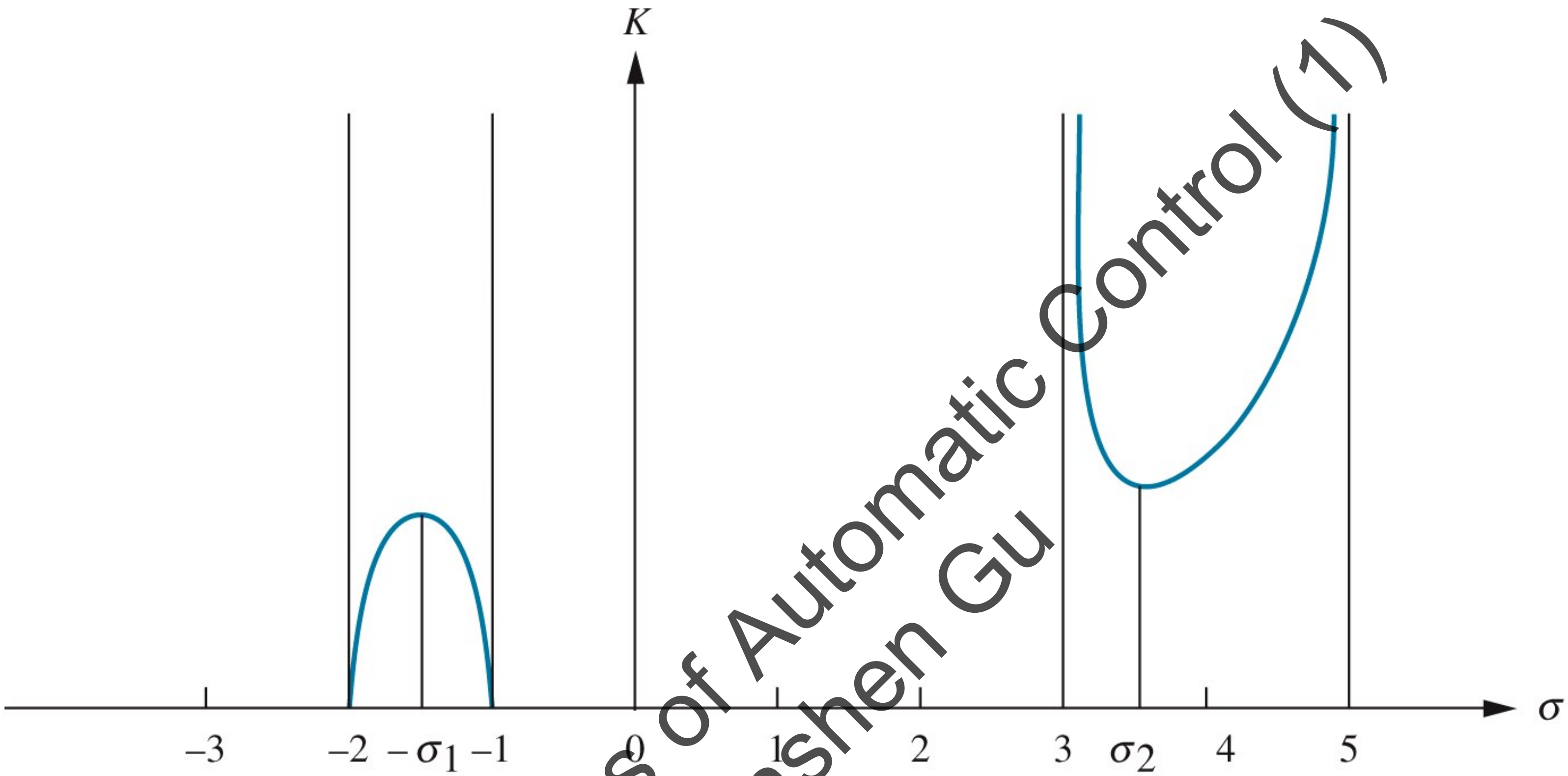


Figure 8.14

The breakaway point is found at the maximum gain between -1 and -2, and the break-in point is found at the minimum gain between +3 and +5.

Method 1: Using differential calculus

dK/ds=0

Example 8.3

Breakaway and Break-in Points via Differentiation

PROBLEM: Find the breakaway and break-in points for the root locus of Figure 8.13, using differential calculus.

SOLUTION: Using the open-loop poles and zeros, we represent the open-loop system whose root locus is shown in Figure 8.13 as follows:

$$KG(s)H(s) = \frac{K(s-3)(s-5)}{(s+1)(s+2)} = \frac{K(s^2-8s+15)}{(s^2+3s+2)} \quad (8.33)$$

But for all points along the root locus, $KG(s)H(s) = -1$, and along the real axis, $s = \sigma$. Hence,

$$\frac{K(\sigma^2-8\sigma+15)}{(\sigma^2+3\sigma+2)} = -1 \quad (8.34)$$

Solving for K, we find

$$K = \frac{-(\sigma^2+3\sigma+2)}{(\sigma^2-8\sigma+15)} \quad (8.35)$$

Differentiating K with respect to σ and setting the derivative equal to zero yields

$$\frac{dK}{d\sigma} = \frac{(11\sigma^2-26\sigma-61)}{(\sigma^2-8\sigma+15)^2} = 0 \quad (8.36)$$

Solving for σ, we find $\sigma = -1.45$ and 3.82, which are the breakaway and break-in points.

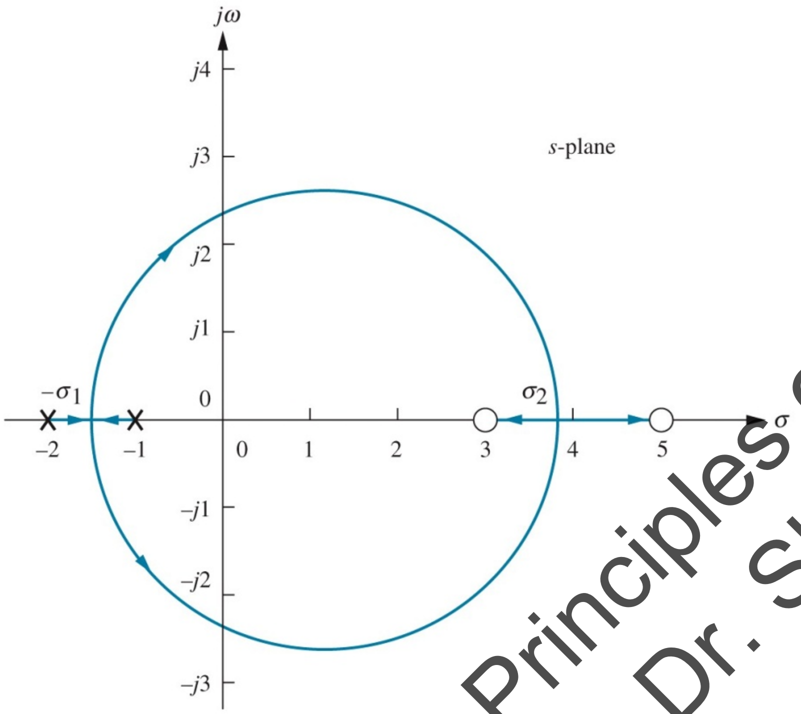


Figure 8.13

Method 2: Transition method

$$\sum_{1}^{m}\frac{1}{\sigma - z_i} = \sum_{1}^{n}\frac{1}{\sigma - p_i}$$

where z_i and p_i are the zeros and poles, respectively, of *G(s)H(s)*. Solving this equation for *s*, the real-axis values that minimize or maximize *K*, yields the breakaway and break-in points without differentiating.

Example 8.4

Breakaway and Break-in Points Without Differentiation

PROBLEM: Repeat Example 8.3 without differentiating.

SOLUTION: Using Eq. (8.37),

$$\frac{1}{\sigma - 3} + \frac{1}{\sigma - 5} = \frac{1}{\sigma + 1} + \frac{1}{\sigma + 2} \qquad (8.38)$$

Simplifying,

$$11\sigma^2 - 26\sigma - 61 = 0 \qquad (8.39)$$

Hence, $\sigma = -1.45$ and 3.82, which agrees with Example 8.3.

Method 3: Computer Program

Simply use the program to search for the point of maximum gain between -1 and -2 and to search for the point of minimum gain between +3 and +5.

TABLE 8.2 Data for breakaway and break-in points for the root locus of Figure 8.13

Real-axis value	Gain		Comment
−1.41	0.008557		
−1.42	0.008585		
−1.43	0.008605		
−1.44	0.008617		
−1.45	0.008623	←	Max. gain: breakaway
−1.46	0.008622		
3.3	44.686		
3.4	37.125		
3.5	33.000		
3.6	30.667		
3.7	29.440		
3.8	29.000	←	Min. gain: break-in
3.9	29.202		

The $j\omega$-Axis Crossings

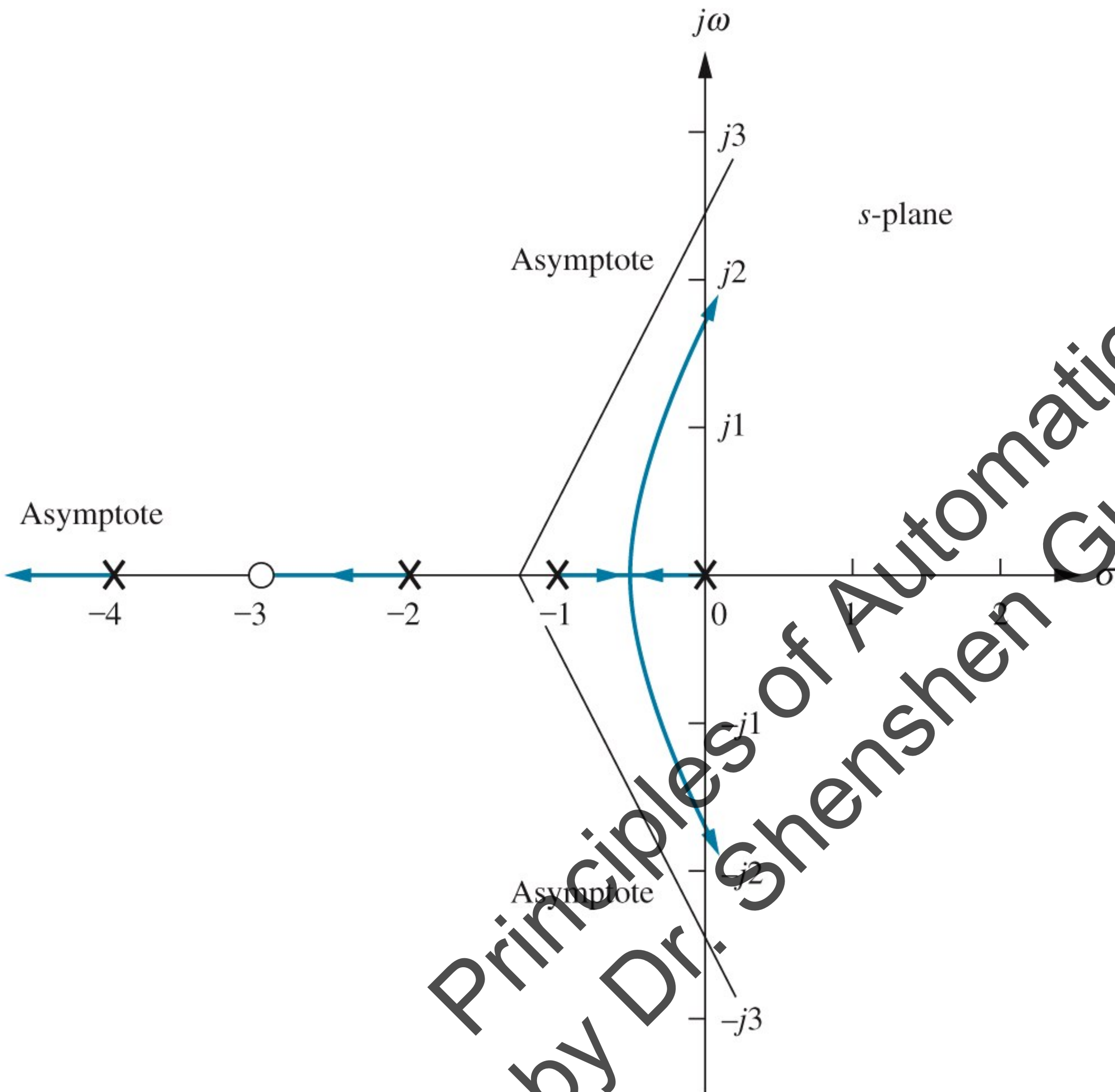


Figure 8.12

Use Routh-Hurwitz criterion

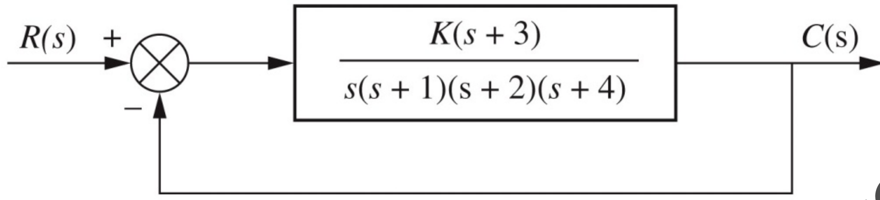


Figure 8.11

Example 8.5

Frequency and Gain at Imaginary-Axis Crossing

PROBLEM: For the system of Figure 8.11, find the frequency and gain, K, for which the root locus crosses the imaginary axis. For what range of K is the system stable?

SOLUTION: The closed-loop transfer function for the system of Figure 8.11 is

$$T(s) = \frac{K(s+3)}{s^4 + 7s^3 + 14s^2 + (8+K)s + 3K} \tag{8.40}$$

Using the denominator and simplifying some of the entries by multiplying any row by a constant, we obtain the Routh array shown in Table 8.3.

A complete row of zeros yields the possibility for imaginary axis roots. For positive values of gain, those for which the root locus is plotted, only the s^1 row can yield a row of zeros. Thus,

$$-K^2 - 65K + 720 = 0 \tag{8.41}$$

From this equation K is evaluated as

$$K = 9.65 \tag{8.42}$$

TABLE 8.3 Routh table for Eq. (8.40)

s^4	1	14	$3K$
s^3	7	$8+K$	
s^2	$90-K$	$21K$	
s^1	$\dfrac{-K^2-65K+720}{90-K}$		
s^0	$21K$		

Forming the even polynomial by using the s^2 row with $K = 9.65$, we obtain

$$(90-K)s^2 + 21K = 80.35s^2 + 202.7 = 0 \tag{8.43}$$

and s is found to be equal to $\pm j1.59$. Thus the root locus crosses the $j\omega$-axis at $\pm j1.59$ at a gain of 9.65. We conclude that the system is stable for $0 \leq K < 9.65$.

Angles of Departure and Arrival

Angle of departure

$$\theta_1 = \theta_2 + \theta_3 - \theta_4 - \theta_5 + \theta_6 - (2k+1)180°$$

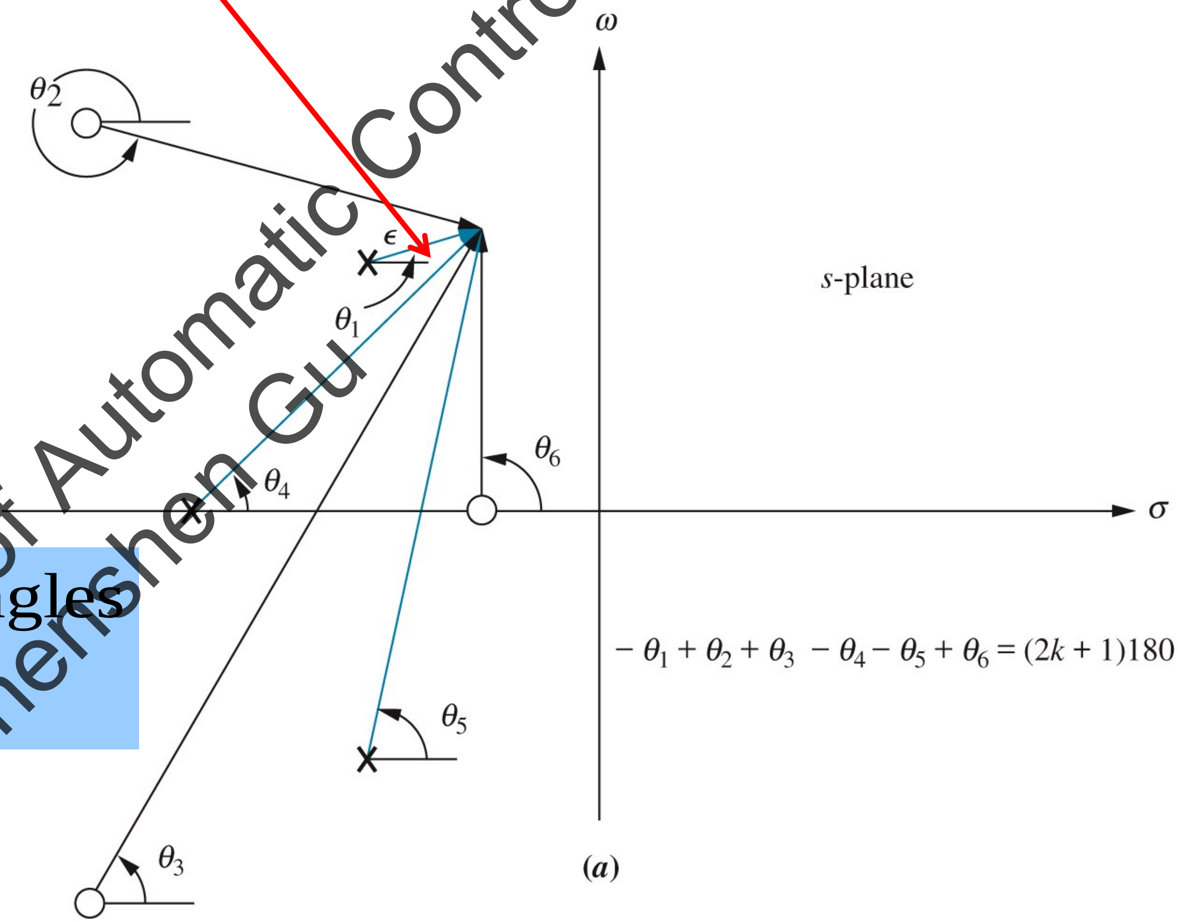


Figure 8.15a
©1992 AIAA

Angle equation (AE)

$$\angle KG(s)H(s) = (2k+1)180°$$

$$\sum \text{zero angles} - \sum \text{pole angles} = (2k+1)180°$$

Angle of arrival

$$\theta_2 = \theta_1 - \theta_3 + \theta_4 + \theta_5 - \theta_6 + (2k+1)180^\circ$$

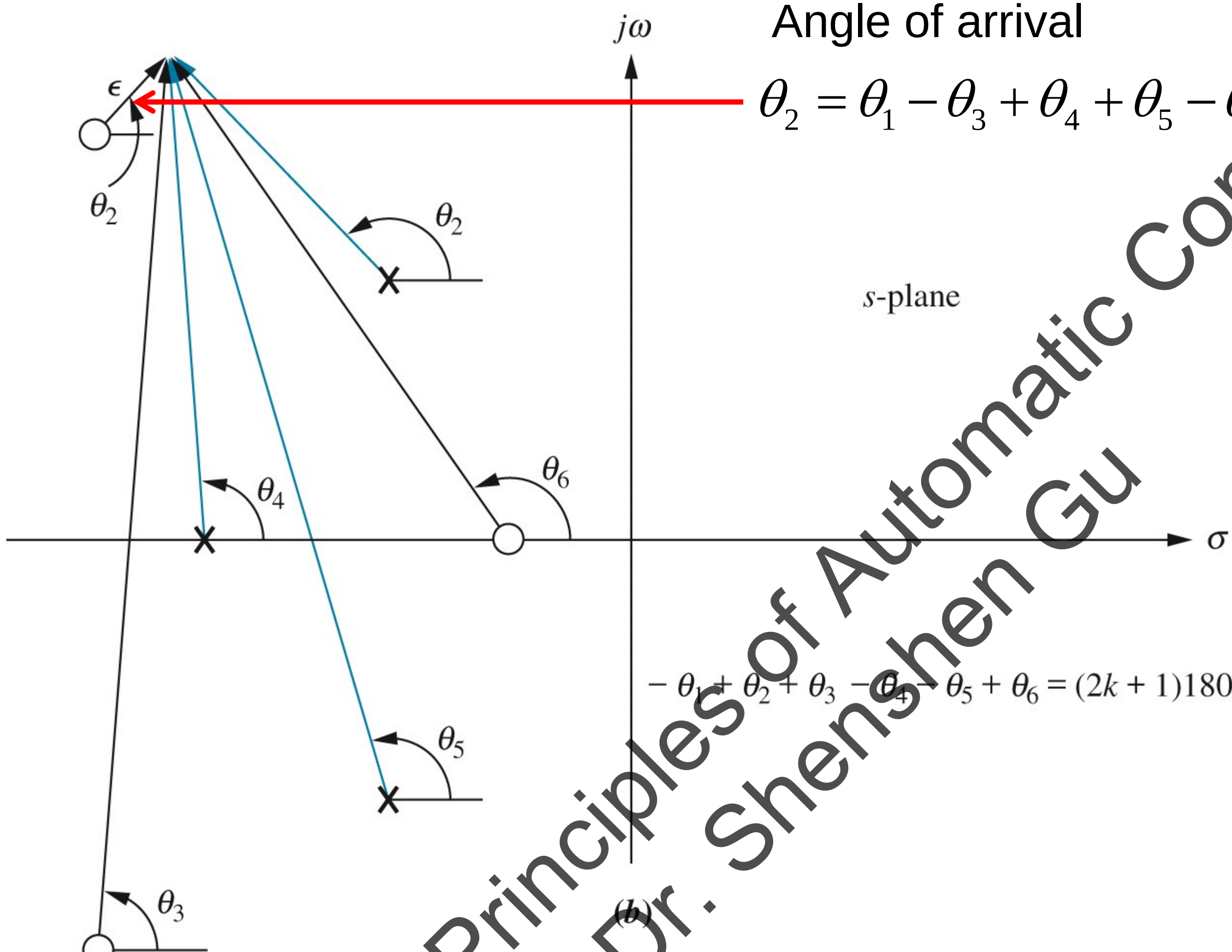


Figure 8.15b

Example 8.6

Angle of Departure from a Complex Pole

PROBLEM: Given the unity feedback system of Figure 8.16, find the angle of departure from the complex poles and sketch the root locus.

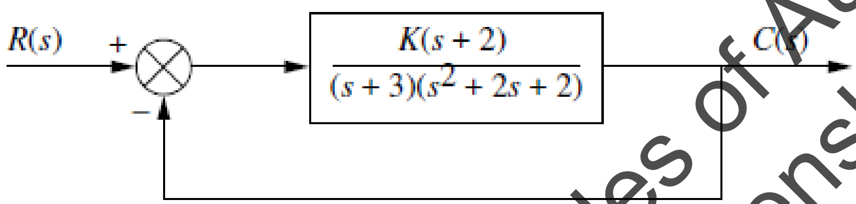


FIGURE 8.16 Unity feedback system with complex poles

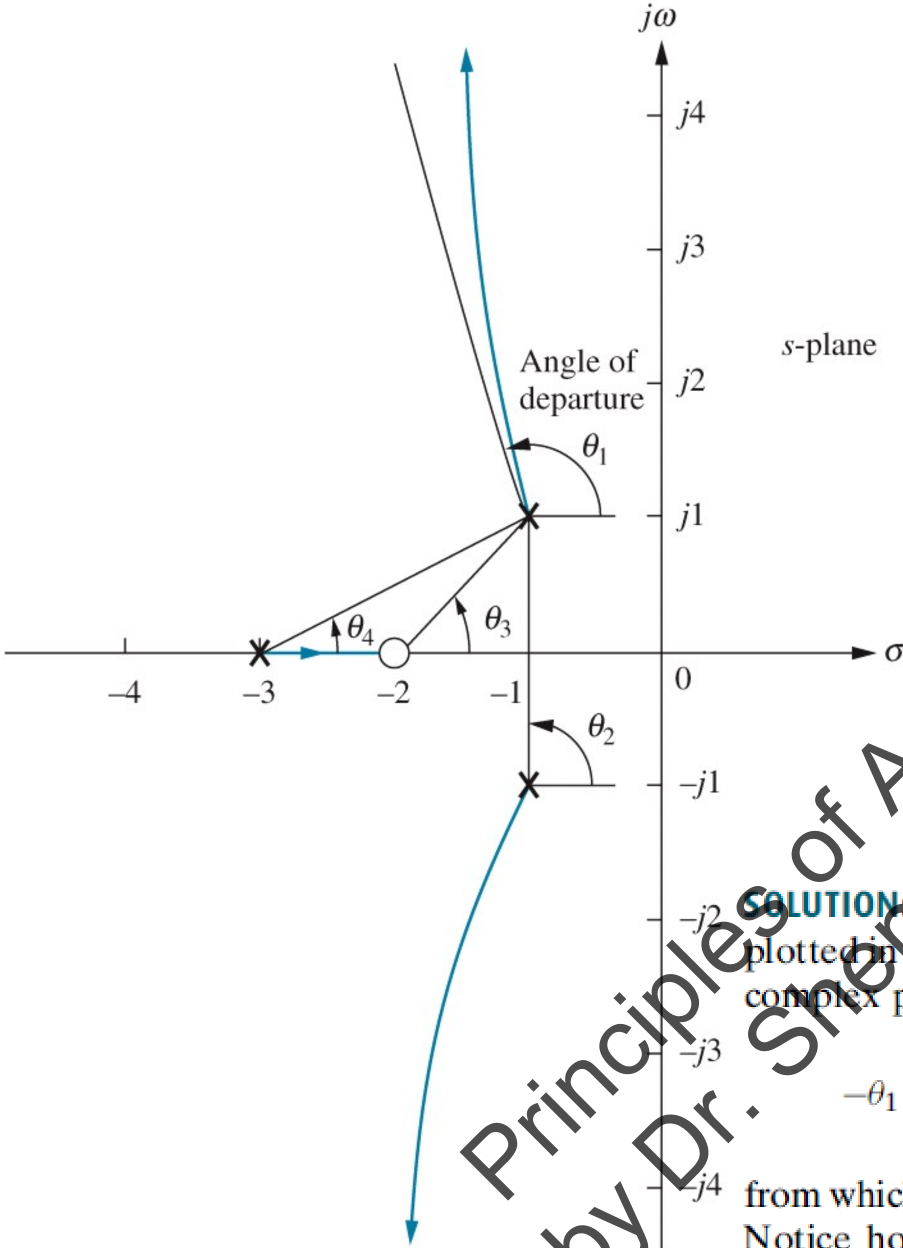


Figure 8.17

SOLUTION: Using the poles and zeros of $G(s) = (s+2)/[(s+3)(s^2+2s+2)]$ as plotted in Figure 8.17, we calculate the sum of angles drawn to a point ϵ close to the complex pole, $-1+j1$, in the second quadrant. Thus,

$$-\theta_1 - \theta_2 + \theta_3 - \theta_4 = -\theta_1 - 90^\circ + \tan^{-1}\left(\frac{1}{1}\right) - \tan^{-1}\left(\frac{1}{2}\right) = 180^\circ \qquad (8.46)$$

from which $\theta = -251.6^\circ = 108.4^\circ$. A sketch of the root locus is shown in Figure 8.17. Notice how the departure angle from the complex poles helps us to refine the shape.

Plotting and Calibrating the Root Locus

Radius r	Angle $\angle$ (degrees)
0.5	–158.4
0.747	–180.0
1.0	–199.9
1.5	–230.4
2.0	–251.5

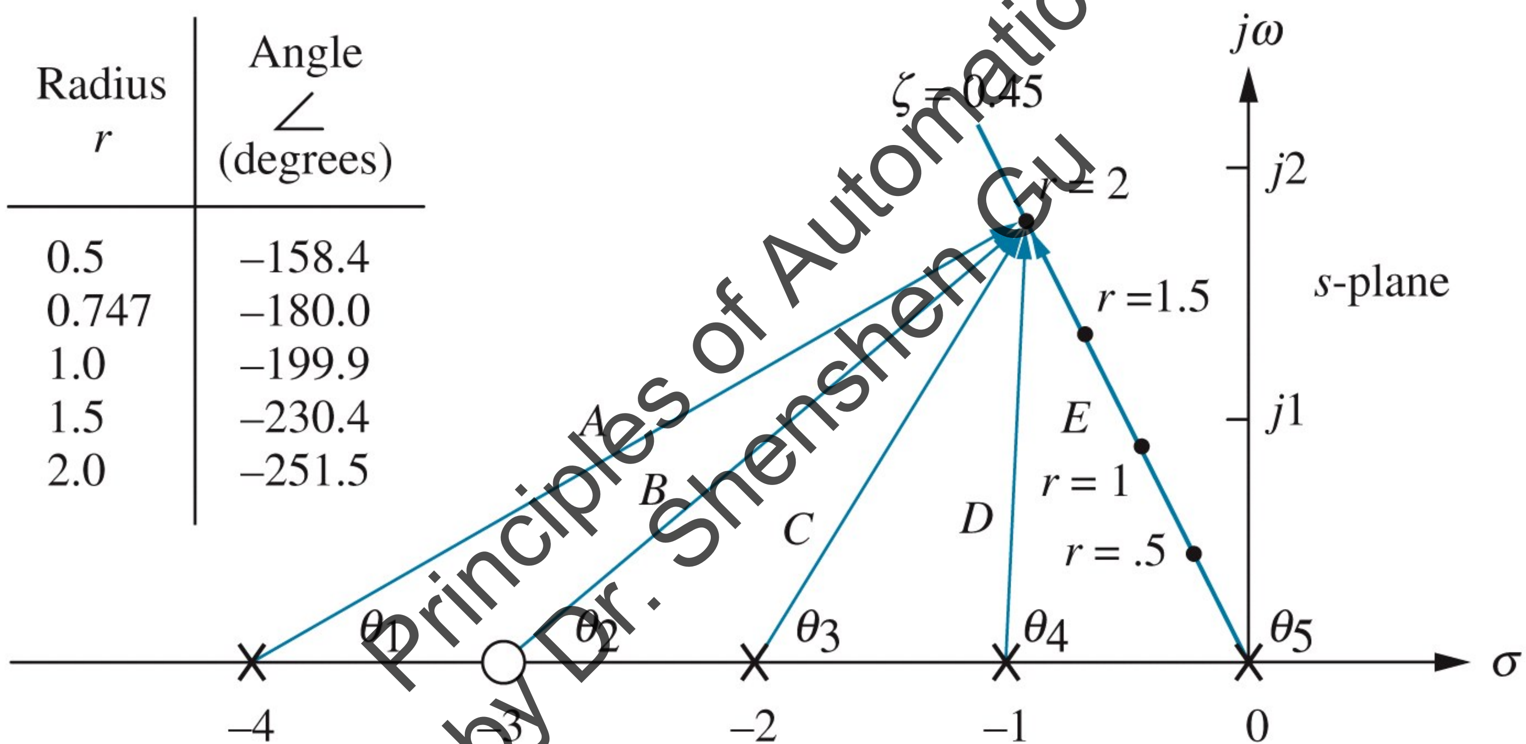


Figure 8.18

Skill-Assessment Exercise 8.4

PROBLEM: Given a unity feedback system that has the forward transfer function

$$G(s) = \frac{K(s+2)}{(s^2 - 4s + 13)}$$

do the following:

a. Sketch the root locus.

b. Find the imaginary-axis crossing.

c. Find the gain, K, at the $j\omega$-axis crossing.

d. Find the break-in point.

e. Find the angle of departure from the complex poles.

ANSWERS:

a. See solution at www.wiley.com/college/nise.

b. $s = \pm j\sqrt{21}$

c. $K = 4$

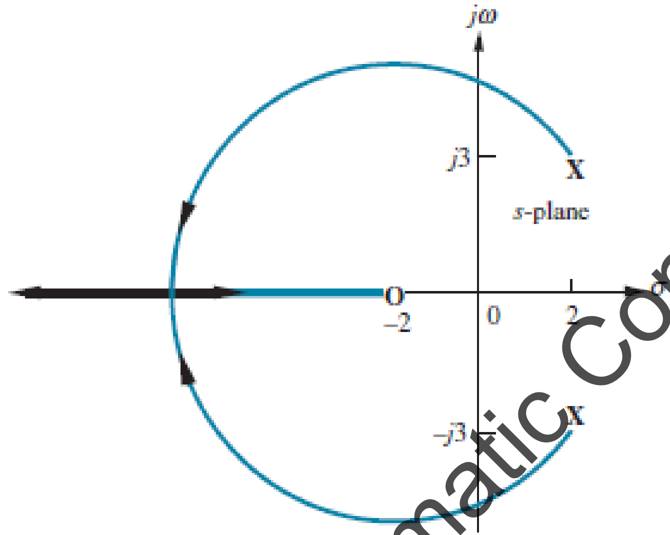
d. Break-in point $= -7$

e. Angle of departure $= -233.1^\circ$

The complete solution is at www.wiley.com/college/nise.

8.4

a.



b. Using the Routh-Hurwitz criteria, we first find the closed-loop transfer function.

$$T(s) = \frac{G(s)}{1+G(s)} = \frac{K(s+2)}{s^2+(K-4)s+(2K+13)}$$

Using the denominator of $T(s)$, make a Routh table.

s^2	1	$2K+13$
s^1	$K-4$	0
s^0	$2K+13$	0

We get a row of zeros for $K = 4$. From the s^2 row with $K = 4$, $s^2 + 21 = 0$. From which we evaluate the imaginary axis crossing at $\sqrt{21}$.

c. From part (b), $K = 4$.

d. Searching for the minimum gain to the left of –2 on the real axis yields –7 at a gain of 18. Thus the break-in point is at –7.

An Example

Basic Rules for Sketching the Root Locus:

- Number of branches: The number of branches of the root locus equals the number of closed-loop poles.
- Symmetry: The root locus is symmetrical about the real axis.
- Real-axis segments: On the real axis, for $K > 0$ the root locus exists to the left of an odd number of real-axis, finite open-loop poles and/or finite open-loop zeros.
- Starting and ending points: The root locus begins at the finite and infinite poles of $G(s)H(s)$ and ends at the finite and infinite zeros of $G(s)H(s)$.

- Behavior at infinity: The root locus approaches straight lines as asymptotes as the locus approaches infinity. Further, the equations of the asymptotes are given by the real-axis intercept and angle in radians as follows:

$$\sigma_a = \frac{\sum \text{finite poles} - \sum \text{finite zeros}}{\#\text{finite poles} - \#\text{finite zeros}}$$

$$\theta_a = \frac{(2k+1)\pi}{\#\text{finite poles} - \#\text{finite zeros}},$$
$$k = 0, 1, 2, \ldots, n-m-1$$

- Additional Rules for Refining the Sketch
  - Real-Axis Breakaway and Break-In Points: Three methods
  - The *jω*-Axis Crossings: Routh-Hurwitz criterion
  - Angles of Departure and Arrival: Angle equation
  - Plotting and Calibrating the Root Locus: Magnitude equation

Example 8.7

Sketching a Root Locus and Finding Critical Points

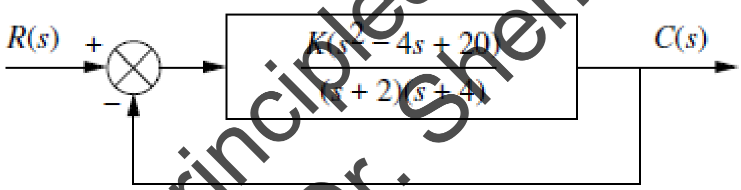
PROBLEM: Sketch the root locus for the system shown in Figure 8.19(a) and find the following:

a. The exact point and gain where the locus crosses the 0.45 damping ratio line

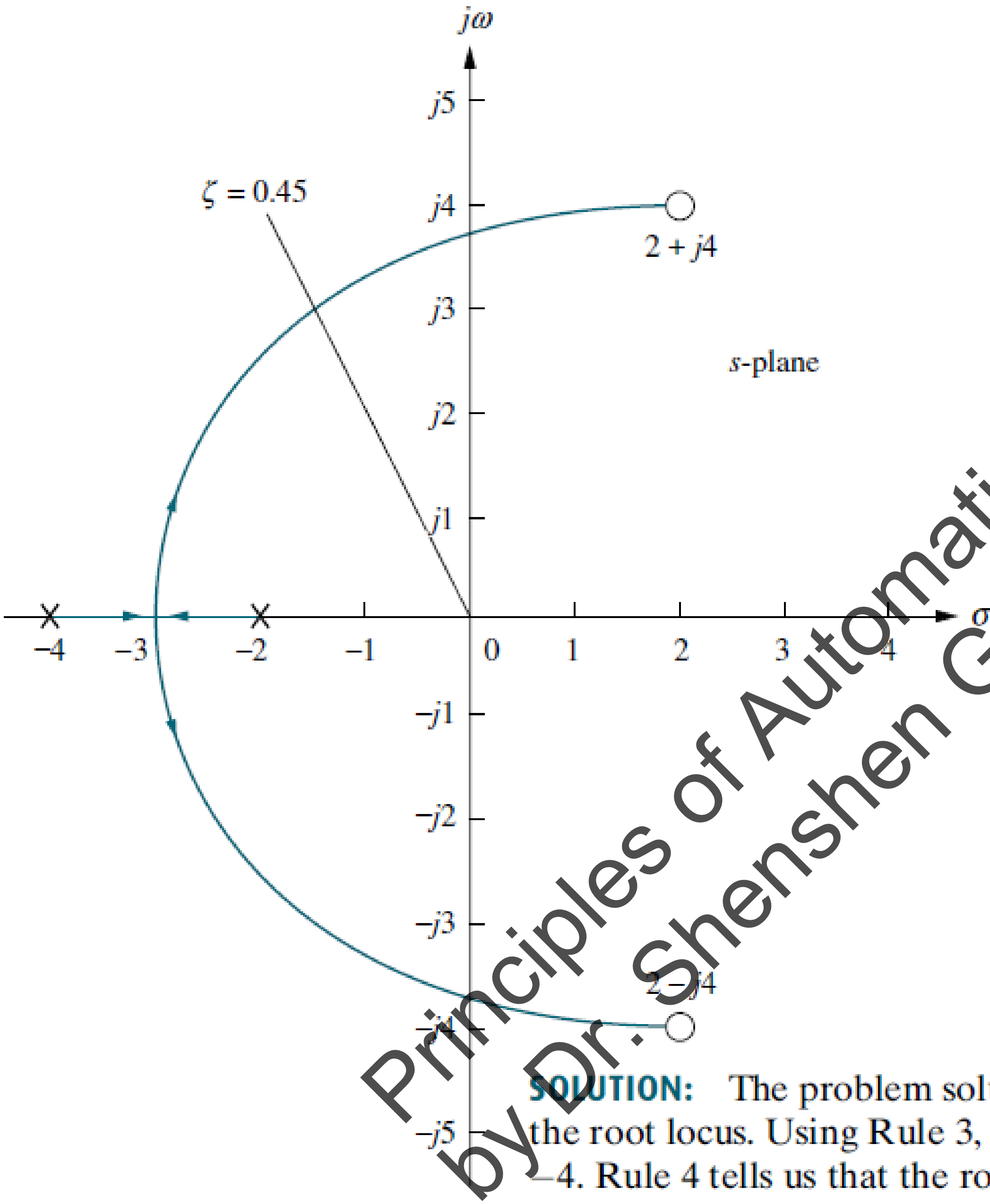
b. The exact point and gain where the locus crosses the $j\omega$-axis

c. The breakaway point on the real axis

d. The range of K within which the system is stable



(a)



(b)

SOLUTION: The problem solution is shown, in part, in Figure 8.19(*b*). First sketch the root locus. Using Rule 3, the real-axis segment is found to be between −2 and −4. Rule 4 tells us that the root locus starts at the open-loop poles and ends at the open-loop zeros. These two rules alone give us the general shape of the root locus.

a. To find the exact point where the locus crosses the $\zeta = 0.45$ line, we can use the root locus program discussed in Appendix H.2 at www.wiley.com/college/nise to search along the line

$$\theta = 180^\circ - \cos^{-1} 0.45 = 116.7^\circ \tag{8.52}$$

for the point where the angles add up to an odd multiple of 180°. Searching in polar coordinates, we find that the root locus crosses the $\zeta = 0.45$ line at $3.4\angle 116.7^\circ$ with a gain, K, of 0.417.

b. To find the exact point where the locus crosses the $j\omega$-axis, use the root locus program to search along the line

$$\theta = 90^\circ \tag{8.53}$$

for the point where the angles add up to an odd multiple of 180°. Searching in polar coordinates, we find that the root locus crosses the $j\omega$-axis at $\pm j3.9$ with a gain of $K = 1.5$.

c. To find the breakaway point, use the root locus program to search the real axis between -2 and -4 for the point that yields maximum gain. Naturally, all points will have the sum of their angles equal to an odd multiple of 180°. A maximum gain of 0.0248 is found at the point -2.88. Therefore, the breakaway point is between the open-loop poles on the real axis at -2.88.

d. From the answer to **b**, the system is stable for K between 0 and 1.5.

Skill-Assessment Exercise 8.5

WileyPLUS
WPCS
Control Solutions

PROBLEM: Given a unity feedback system that has the forward transfer function

$$G(s) = \frac{K(s-2)(s-4)}{(s^2+6s+25)}$$

do the following:

a. Sketch the root locus.

b. Find the imaginary-axis crossing.

c. Find the gain, K, at the $j\omega$-axis crossing.

d. Find the break-in point.

e. Find the point where the locus crosses the 0.5 damping ratio line.

f. Find the gain at the point where the locus crosses the 0.5 damping ratio line.

g. Find the range of gain, K, for which the system is stable.

ANSWERS:

a. See solution at www.wiley.com/college/nise.

b. $s = \pm j4.06$

c. $K = 1$

d. Break-in point $= +2.89$

e. $s = -2.42 + j4.18$

f. $K = 0.108$

g. $K < 1$

The complete solution is at www.wiley.com/college/nise.

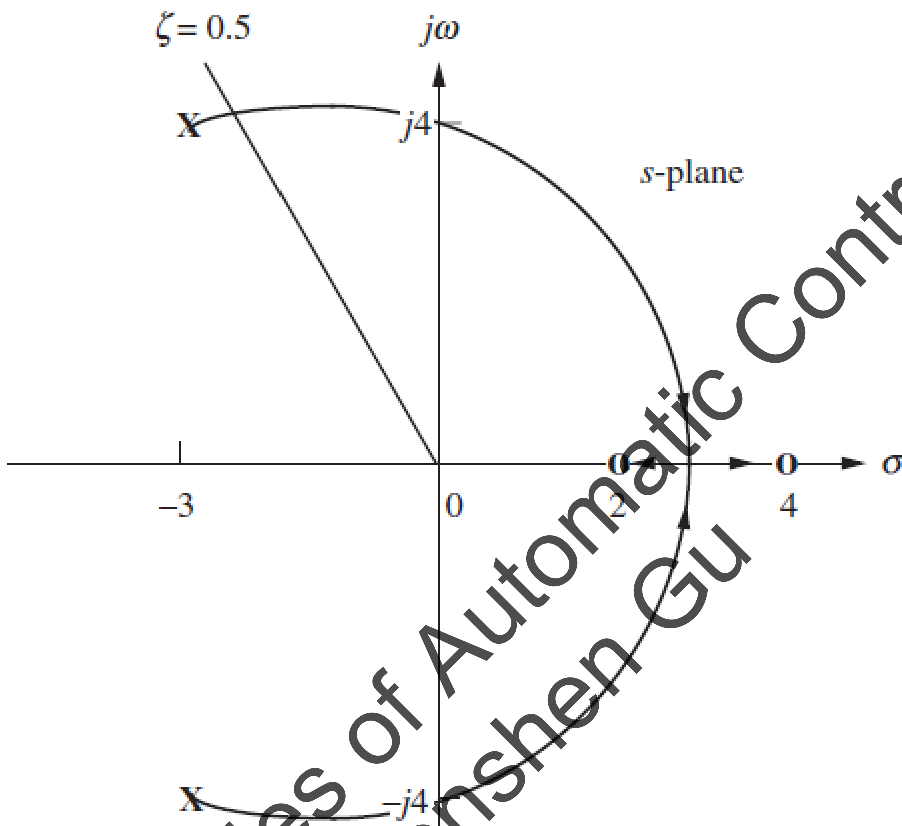
TryIt 8.3

Use MATLAB, the Control System Toolbox, and the following statements to plot the root locus for Skill-Assessment Exercise 8.5. Solve the remaining parts of the problem by clicking on the appropriate points on the plotted root locus.

```
numg=poly([2 4]);
deng=[1 6 25];
G=tf(numg,deng)
rlocus(G)
z=0.5
sgrid(z,0)
```

8.5

a.



b. Search along the imaginary axis and find the 180° point at $s = \pm j4.06$.

c. For the result in part (b), $K = 1$.

d. Searching between 2 and 4 on the real axis for the minimum gain yields the break-in at $s = 2.89$.

e. Searching along $\zeta = 0.5$ for the 180° point we find $s = -2.42 + j4.18$.

f. For the result in part (e), $K = 0.108$.

g. Using the result from part (c) and the root locus, $K < 1$.

Summary

- In this chapter, we examined the root locus, a powerful tool for the analysis and design of control systems.
- The root locus empowers us with qualitative and quantitative information about the stability and transient response of feedback control systems.
- The root locus allows us to find the poles of the closed-loop system by starting from the open-loop system's poles and zeros.
- It is basically a graphical root-solving technique.
- We looked at ways to sketch the root locus rapidly, even for higher-order systems.

Summary (Cont.)

- The sketch gave us qualitative information about changes in the transient response as parameters were varied.
- From the locus we were able to determine whether a system was unstable for any range of gain.
- We developed the criterion for determining whether a point in the s-plane was on the root locus: The angles from the open-loop zeros, minus the angles from the open-loop poles drawn to the point in the s-plane, add up to an odd multiple of 180.