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Principles of Automatic Control (1)

自动控制原理1

Topic 3
Time Response
(Chapter 4 in text book)

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New terminologies in this topic

- Pole 极点
- Zero 零点
- System response 系统响应
- Transient response 瞬态响应
- Underdamped 欠阻尼
- Overdamped 过阻尼
- Critically damped 临界阻尼
- Undamped 无阻尼
- Damping ratio 阻尼率
- Natural frequency 自然频率
- Damped frequency 阻尼频率
- Pythagorean theorem 勾股定理
- Settling time 调节时间
- Peak time 峰值时间
- Rise time 上升时间
- Percent overshoot 超调比例
- Qualitative analysis 定性分析
- Reciprocal 倒数
- Oscillation 振荡
- Time constant 时间常数
- Exponential frequency 指数频率
- Slope 斜率
- Complex conjugate 共轭复数
- Parameter 参数

Learning Outcomes for Topic 3

After completing this topic, you will be able to:

- Use poles and zeros of transfer functions to determine the time response of a control system;
- Describe quantitatively the transient response of first-order systems;
- Write the general response of second-order systems given the pole location;
- Find the damping ratio and natural frequency of a second-order system;
- Find the settling time, peak time, percentage overshoot, and rise time for an underdamped second-order system.

Outline

- Brief Introduction
- Poles, Zeros, and System Response
- First-Order Systems
- Second-Order Systems: Introduction
- The General Second-Order Systems
- Underdamped Second-Order Systems

Brief Introduction

- We now know how to obtain the transfer function, we need to study the response of the system to specific inputs (specifically to a step input) to understand the behavior of the system. This will allow us to decide what kind of controller is required to bring the system under control (desired behavior)

Poles, Zeros, and System Response

- Total response = Natural response + Forced response
  - Natural response described the way the system dissipates or acquires energy. It is dependent only on the system, not the input
  - Forced response is dependent on the input.

- Solving a differential equation or taking the inverse Laplace transform can be used to evaluate this output. However, these techniques are laborious and time-comsuming.

- We need a qualitative method to analyze and design very rapidly.
  - The use of poles and zeros and their relationship to the time response.

- The roots of factors on the ***numerator*** of the Transfer Function are called ***Zeros*** and those of the ***denominator*** are called ***Poles***.

$$G(s) = K\frac{(s-z_1)(s-z_2)\cdots(s-z_n)}{(s-p_1)(s-p_2)\cdots(s-p_m)}$$

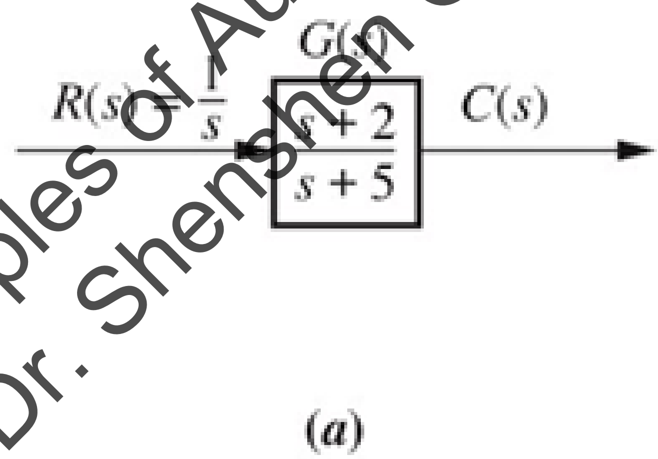
ZEROS

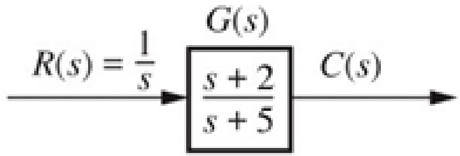
POLES

- Poles and zeros can be complex, e.g.: $p_n = a + jb;\ j = \sqrt{-1}$

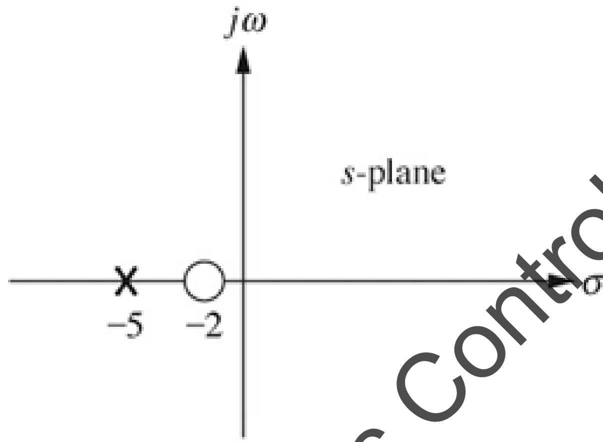
Example

Find c(t) of this system.





(a)



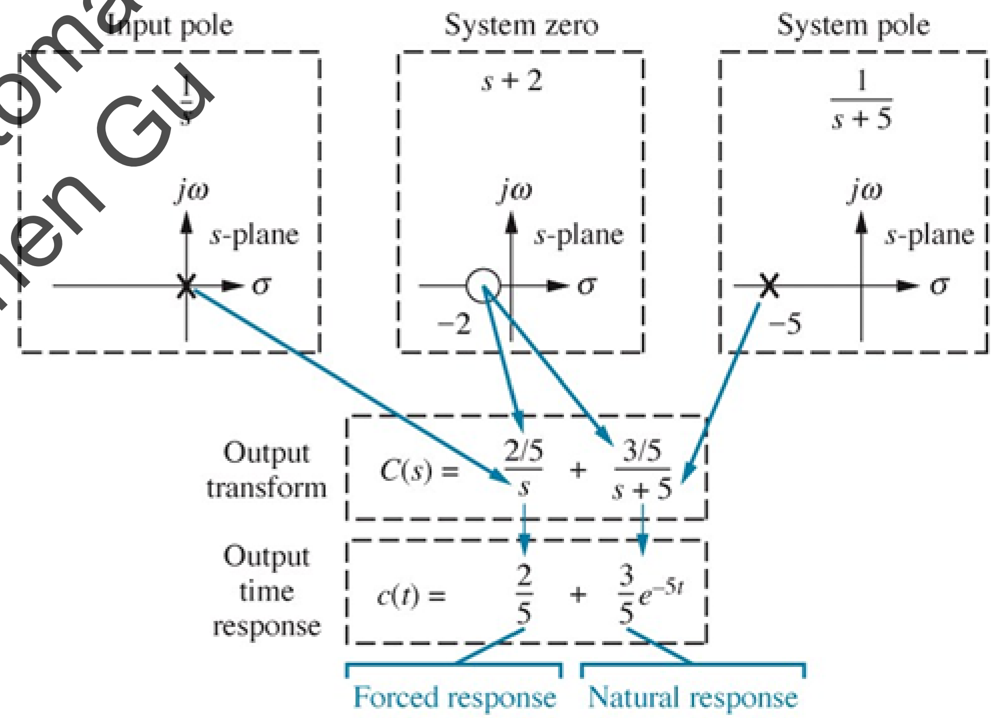
(b)

$$C(s)=R(s)G(s)$$

$$=\frac{(s+2)}{s(s+5)}=\frac{A}{s}+\frac{B}{(s+5)}$$

$$A=\left.\frac{(s+2)}{(s+5)}\right|_{s\to 0}=\frac{2}{5}\qquad B=\left.\frac{(s+2)}{s}\right|_{s\to -5}=\frac{3}{5}$$

$$c(t)=\frac{2}{5}+\frac{3}{5}e^{-5t}$$



(c)

- From this example, we draw the following conclusion:
  - ***A pole of the input function*** generates the form of the ***forced response*** (i.e., the pole at the origin generated a step function at the output);
  - ***A pole of the transfer function*** generates the form of the ***natural response*** (i.e., the pole at -5 generated e^{-5t});
  - A pole on the real axis generates an exponential response of the form e^{-at}, where -a is the pole location on the real axis. Thus, the farther to the left a pole is on the negative real axis, the faster the exponential transient response will decay to zero (again, the pole at -5 generated e^{-5t});
  - ***The zeros and poles*** generate the ***amplitudes*** for both the forced and natural responses.

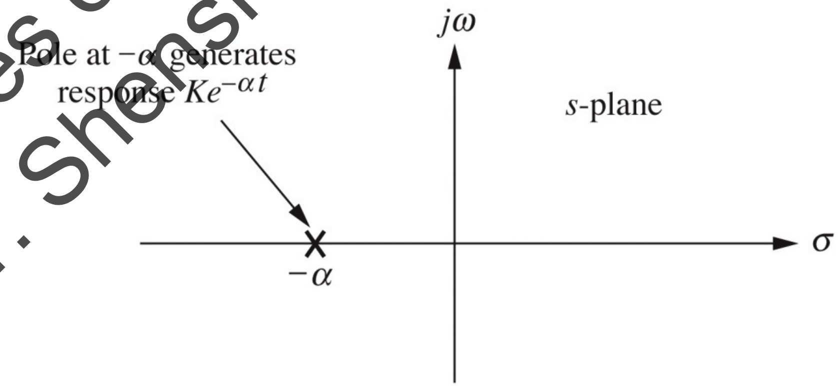


Figure 4.2

Example 4.1

Evaluating Response Using Poles

PROBLEM: Given the system of Figure 4.3, write the output, $c(t)$, in general terms. Specify the forced and natural parts of the solution.

SOLUTION: By inspection, each system pole generates an exponential as part of the natural response. The input's pole generates the forced response. Thus,

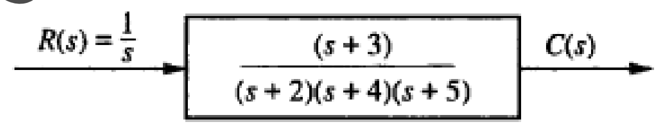


FIGURE 4.3 System for Example 4.1

$$C(s) \equiv \underbrace{\frac{K_1}{s}}_{\text{Forced response}} + \underbrace{\frac{K_2}{s+2} + \frac{K_3}{s+4} + \frac{K_4}{s+5}}_{\text{Natural response}} \tag{4.3}$$

Taking the inverse Laplace transform, we get

$$c(t) \equiv \underbrace{K_1}_{\text{Forced response}} + \underbrace{K_2 e^{-2t} + K_3 e^{-4t} + K_4 e^{-5t}}_{\text{Natural response}} \tag{4.4}$$

Skill-Assessment Exercise 4.1

PROBLEM: A system has a transfer function, $G(s) = \dfrac{10(s+4)(s+6)}{(s+1)(s+7)(s+8)(s+10)}$. Write, by inspection, the output, $c(t)$, in general terms if the input is a unit step.

ANSWER: $c(t) \equiv A + Be^{-t} + Ce^{-7t} + De^{-8t} + Ee^{-10t}$

In this section, we learned that poles determine the nature of the time response: Poles of the input function determine the form of the forced response, and poles of the transfer function determine the form of the natural response. Zeros and poles of the input or transfer function contribute to the amplitudes of the component parts of the total response. Finally, poles on the real axis generate exponential responses.

4.1

For a step input

$$C(s) = \frac{10(s+4)(s+6)}{s(s+1)(s+7)(s+8)(s+10)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+7} + \frac{D}{s+8} + \frac{E}{s+10}$$

Taking the inverse Laplace transform,

$$c(t) = A + Be^{-t} + Ce^{-7t} + De^{-8t} + Ee^{-10t}$$

First-Order Systems

- A first-order system

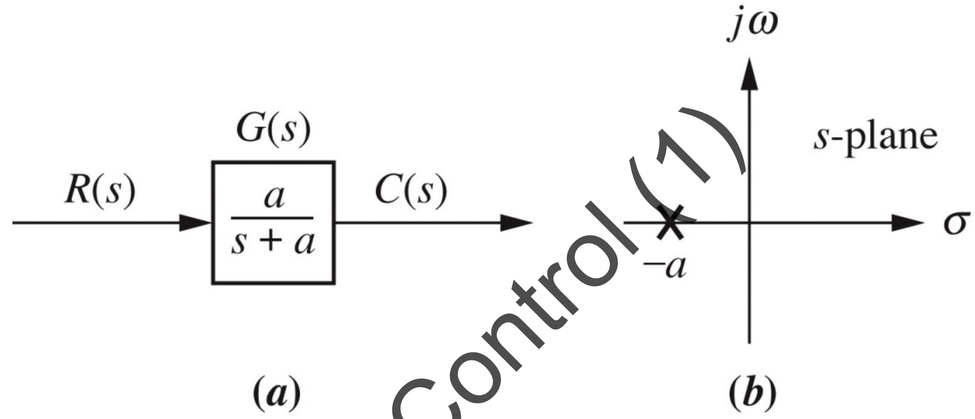


Figure 4.4

- When the input is a unit step, $R(s)=\frac{1}{s}$

$$C(s)=R(s)G(s)=\frac{a}{s(s+a)}$$

- Taking the inverse Laplace transform

$$c(t)=c_f(t)+c_n(t)=1-e^{-at}$$

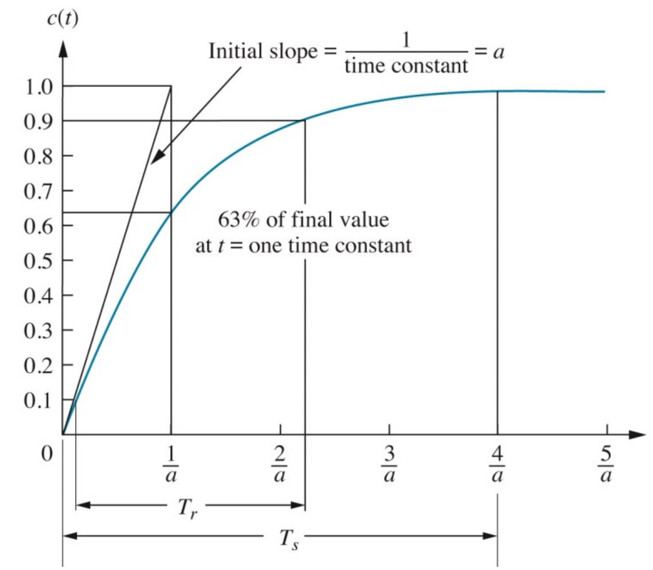


Figure 4.5

- The input pole at the origin -> forced response $c_f(t)=1$
- The system pole at –a -> natural response $c_n(t)=-e^{-at}$

Three transient response performance specifications: Time constant, rise time and settling time

- Let us examine the significance of parameter *a*.
- When $t=1/a$

$$e^{-at}\Big|_{t=1/a} = e^{-1} = 0.37$$

Or

$$c(t)\Big|_{t=1/a} = 1 - e^{-1} = 0.63$$

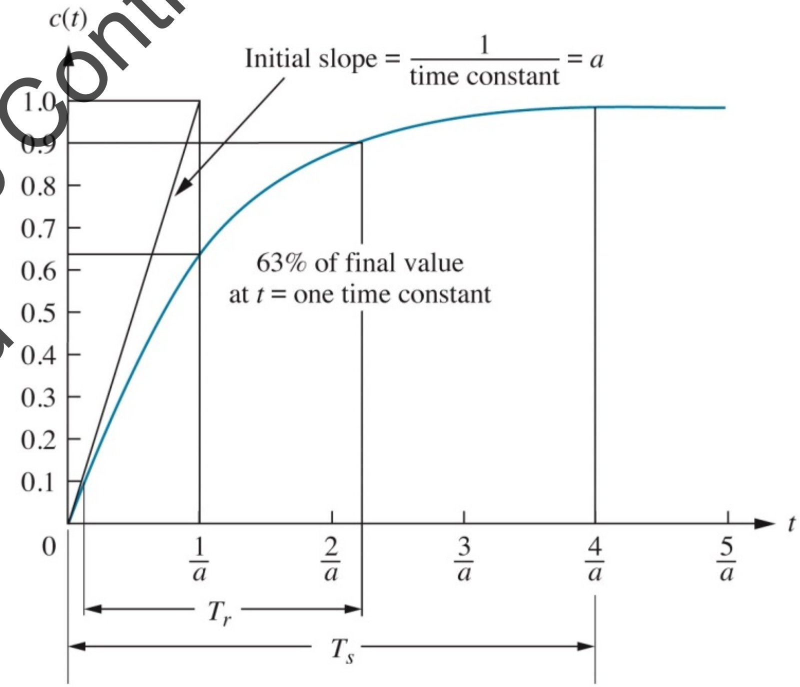


Figure 4.5

We call *1/a* the ***time constant*** of the response.

The time constant is the time it takes for the step response to rise to 63% of its final values

Time constant and exponential frequency

- The reciprocal of the time constant has the unit (1/sec), or frequency.
- We can call the parameter *a* the *exponential frequency*.
- Since the derivative of e^{-at} is $-a$ when $t=0$, a is the initial rate of change of the exponential at $t=0$.

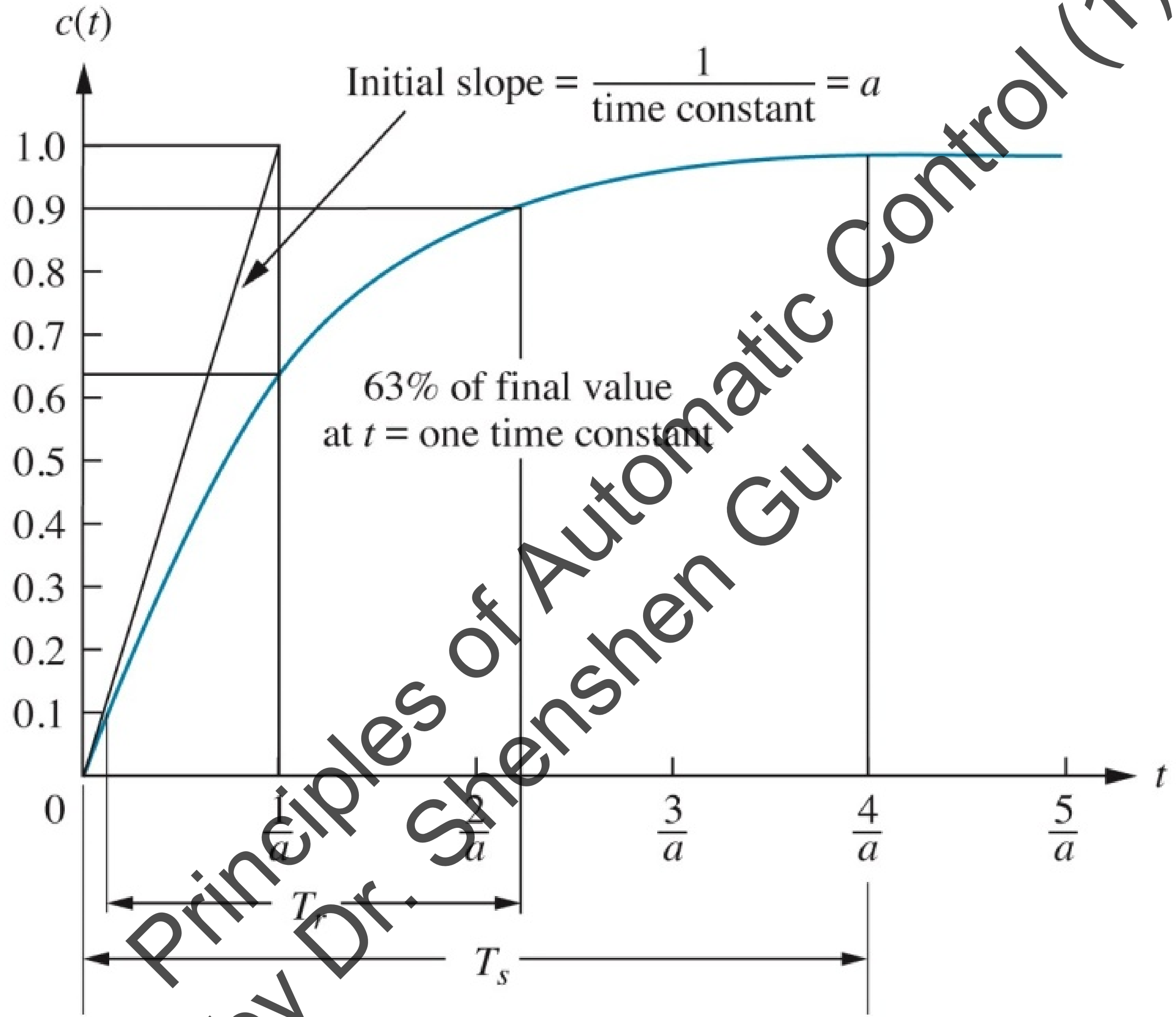

c(t)
Initial slope = 1 / time constant = a
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
63% of final value
at t = one time constant
t
1/a
2/a
3/a
4/a
5/a
T_r
T_s

Figure 4.5

Rise time, T_r

- Rise time is defined as the time for the waveform to go from 0.1 to 0.9 of its final value.

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

$$c(t) = 0.9 \rightarrow t = \frac{2.31}{a}$$

$$c(t) = 0.1 \rightarrow t = \frac{0.11}{a}$$

$$T_r = \frac{2.31}{a} - \frac{0.11}{a} = \frac{2.2}{a}$$

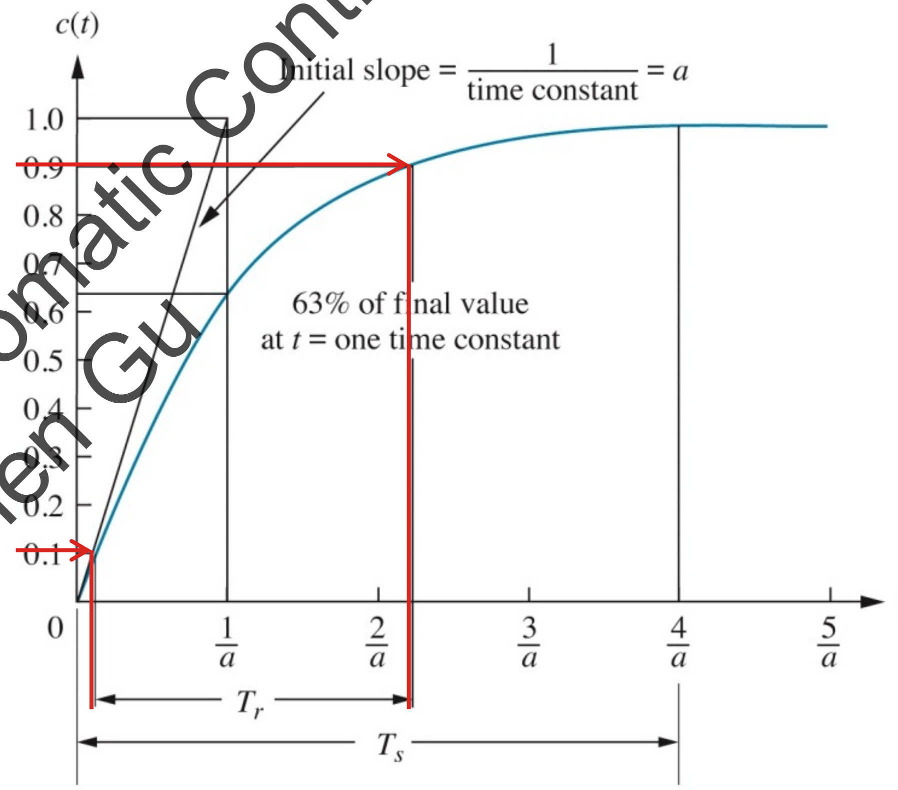


Figure 4.5

Settling Time, T_s

- Settling time is defined as the time for the response to reach, **and stay within**, **2%** of its final value.

$$c(t)=c_f(t)+c_n(t)=1-e^{-at}$$

$$c(t)=0.98$$

$$T_s=\frac{4}{a}$$

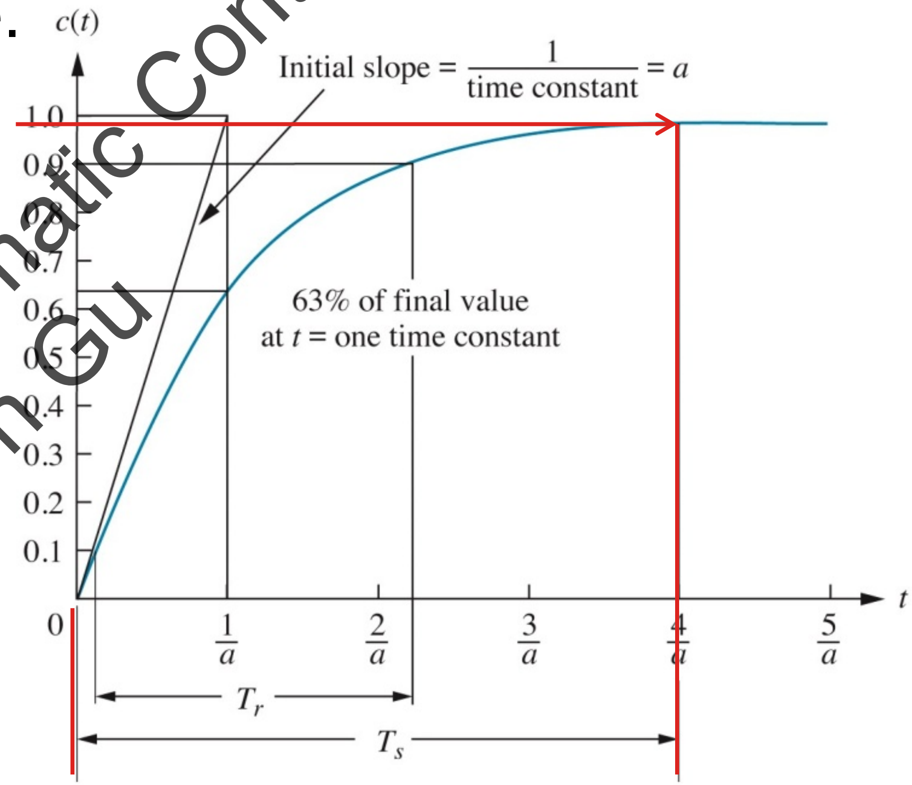


Figure 4.5

First-Order Transfer Functions via Testing

- Often it is not possible or practical to obtain a system's transfer function analytically.
- With a step input, we can measure the time constant and the steady-state value, from which the transfer function can be calculated.
- Consider a simple first-order system, *G(s)=K/(s+a)*, whose step response is

$$C(s)=\frac{K}{s(s+a)}=\frac{K/a}{s}-\frac{K/a}{(s+a)}$$

- If we can identify *K* and *a* from laboratory testing, we can obtain the transfer function of the system.

Example

- We determine that this system has the first-order characteristics such as ***no overshoot*** and ***nonzero initial slope***.
- The time constant is the time it takes for the step response to rise to *63%* of its final values.
- The final value is about *0.72*, the time constant is evaluated where the curve reaches *0.63×0.72=0.45*, or about *0.13* second.

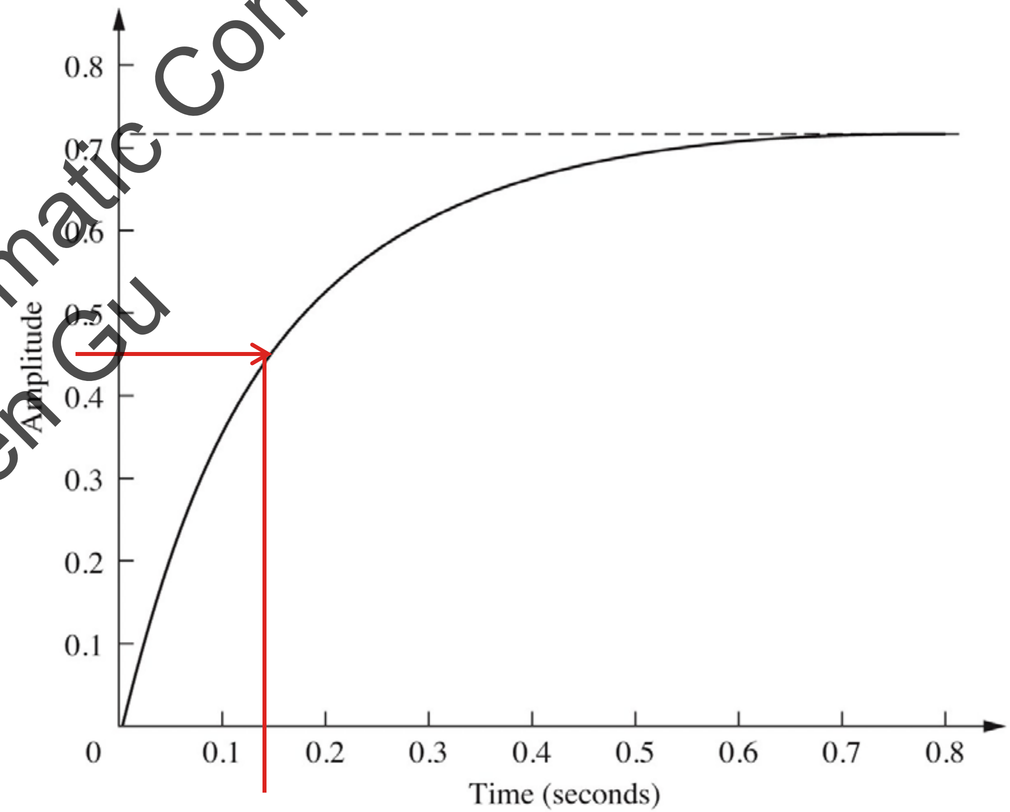


Figure 4.6

Example

- Hence, *a=1/0.13=7.7*.

$$C(s)=\frac{K}{s(s+a)}=\frac{K/a}{s}-\frac{K/a}{(s+a)}$$

- To find K, we realized that the forced response reaches a steady-state value of K/a=0.72. We get K=5.54.

$$G(s)=\frac{5.54}{(s+7.7)}$$

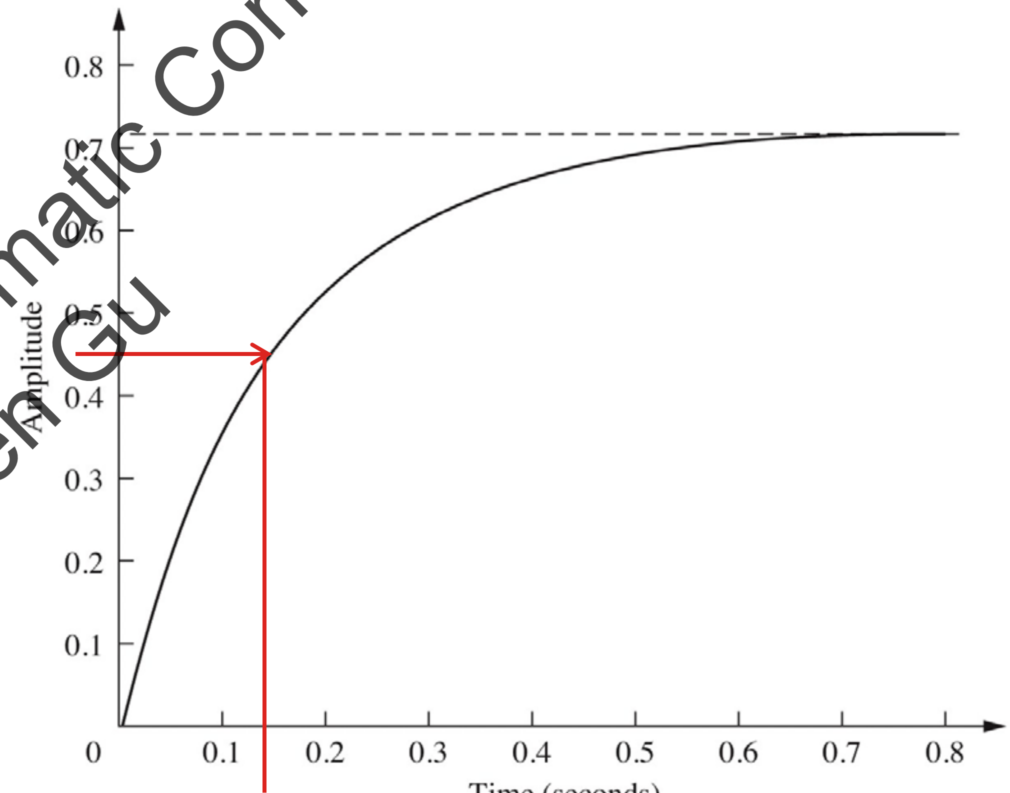


Figure 4.6

Skill-Assessment Exercise 4.2

PROBLEM: A system has a transfer function, $G(s) = \frac{50}{s+50}$. Find the time constant, T_c, settling time, T_s, and rise time, T_r.

ANSWER: $T_c = 0.02$ s, $T_s = 0.08$ s, and $T_r = 0.044$ s.

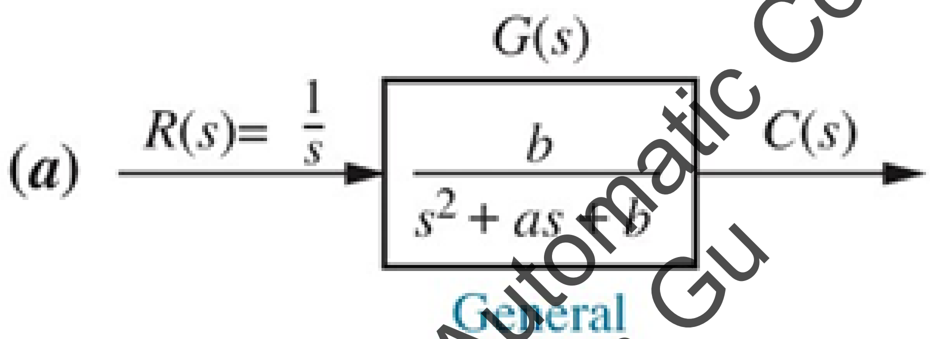
The complete solution is located at www.wiley.com/college/nise.

4.2

Since $a = 50$, $T_c = \frac{1}{a} = \frac{1}{50} = 0.02\,\text{s}$; $T_s = \frac{4}{a} = \frac{4}{50} = 0.08\,\text{s}$; and $T_r = \frac{2.2}{a} = \frac{2.2}{50} = 0.044\,\text{s}$.

Second-Order Systems: Introduction

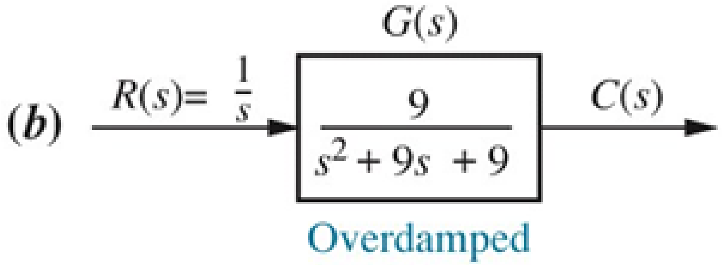
- First order systems have only a *single dominant pole*.

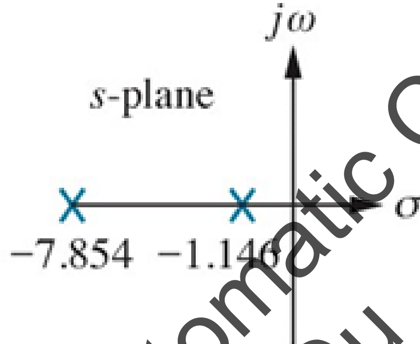


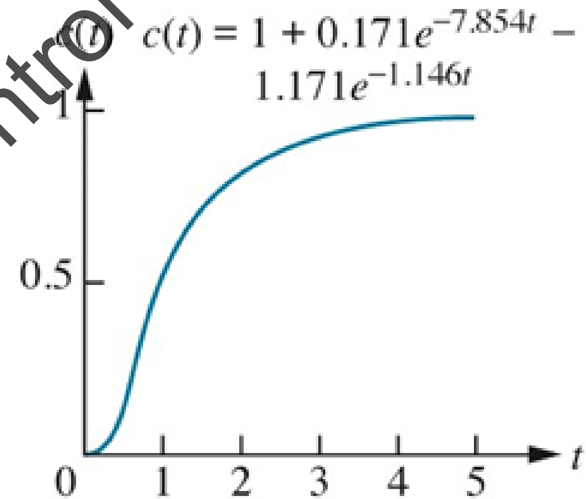
- Second order systems have ***two poles*** affecting the response: (4 possibilities)
  - Two distinct real poles
  - Two complex conjugate poles
  - Two imaginary poles
  - Two repeated real poles

- Changes in the parameters of a second-order system can change the form of the response. A second-order system can display characteristics much like a first-order system, or display damped or pure oscillations for its transient response.

Overdamped Response (Two distinct real poles)





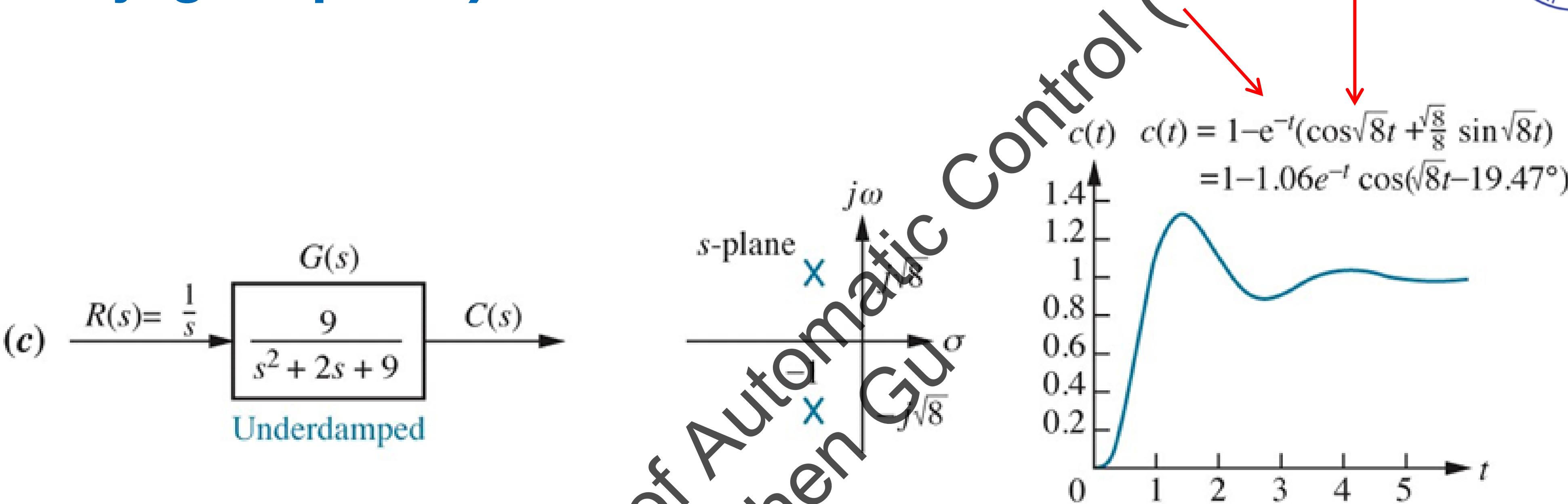


$$C(s)=\frac{9}{s\left(s^2+9s+9\right)}=\frac{9}{s(s+7.854)(s+1.146)}$$

$$c(t)=K_1+K_2e^{-7.854t}+K_3e^{-1.146t}$$

Underdamped Response (Two complex conjugate poles)

$$s = -1 \pm j\sqrt{8}$$



- The real part of the pole matches the exponential decay frequency of the sinusoid's amplitude.
- The imaginary part of the pole matches the frequency of the sinusoidal oscillation.

The steady-state response was generated by the input pole located at the origin

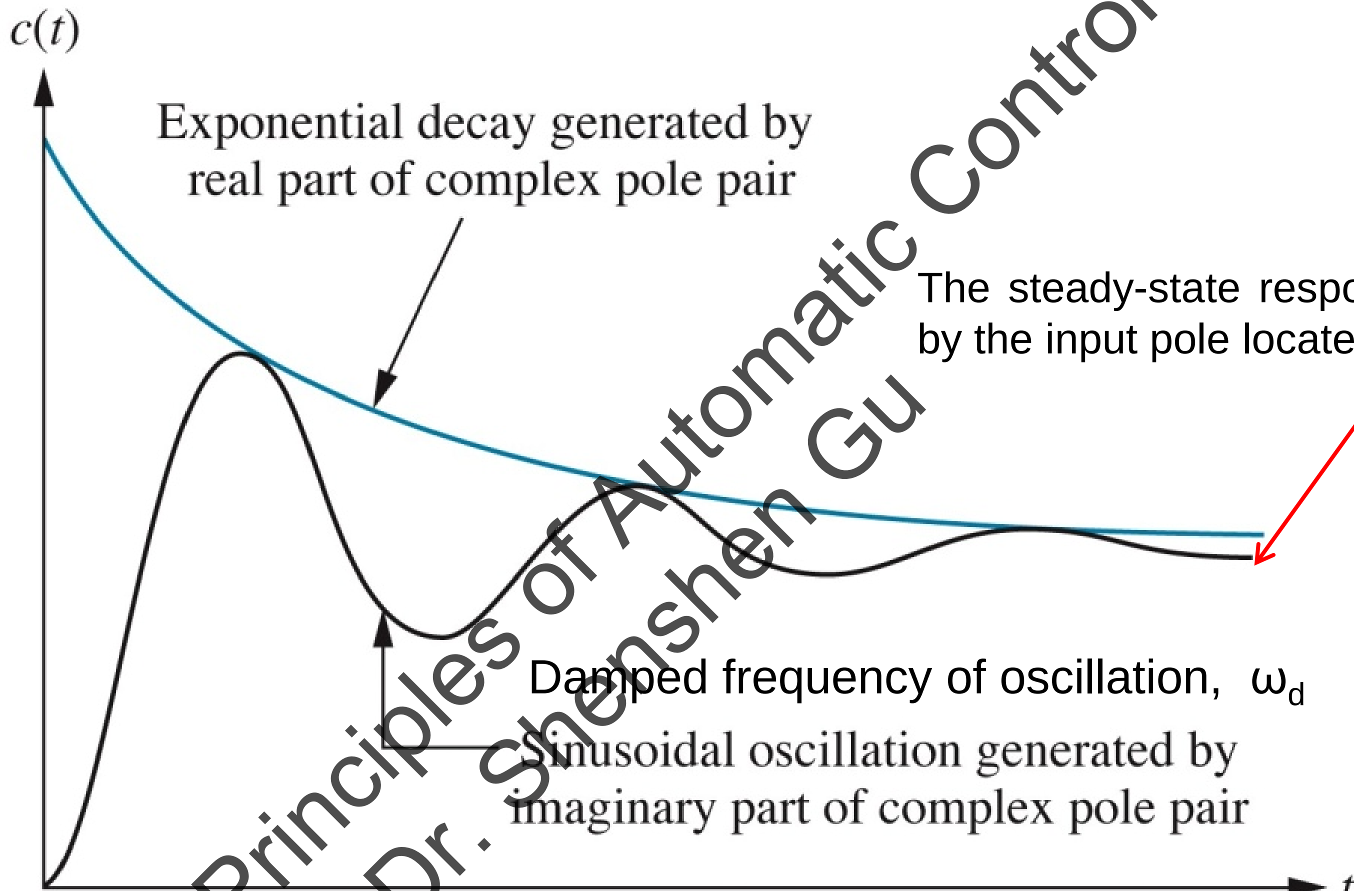


Figure 4.8

Example 4.2

Form of Underdamped Response Using Poles

PROBLEM: By inspection, write the form of the step response of the system in Figure 4.9.

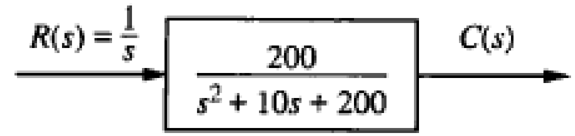
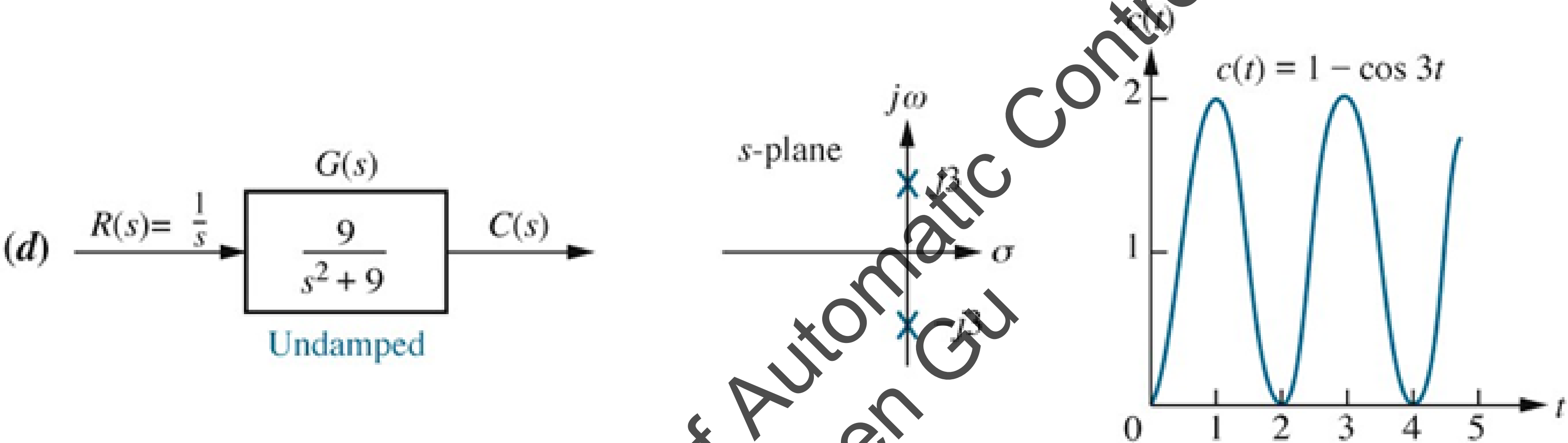


FIGURE 4.9 System for Example 4.2

SOLUTION: First we determine that the form of the forced response is a step. Next we find the form of the natural response. Factoring the denominator of the transfer function in Figure 4.9, we find the poles to be $s = -5 \pm j13.23$. The real part, -5, is the exponential frequency for the damping. It is also the reciprocal of the time constant of the decay of the oscillations. The imaginary part, 13.23, is the radian frequency for the sinusoidal oscillations. Using our previous discussion and Figure 4.7(c) as a guide, we obtain $c(t) = K_1 + e^{-5t}(K_2 \cos 13.23t + K_3 \sin 13.23t) = K_1 + K_4 e^{-5t}(\cos 13.23t - \phi)$, where $\phi = \tan^{-1} K_3/K_2$, $K_4 = \sqrt{K_2^2 + K_3^2}$, and $c(t)$ is a constant plus an exponentially damped sinusoid.

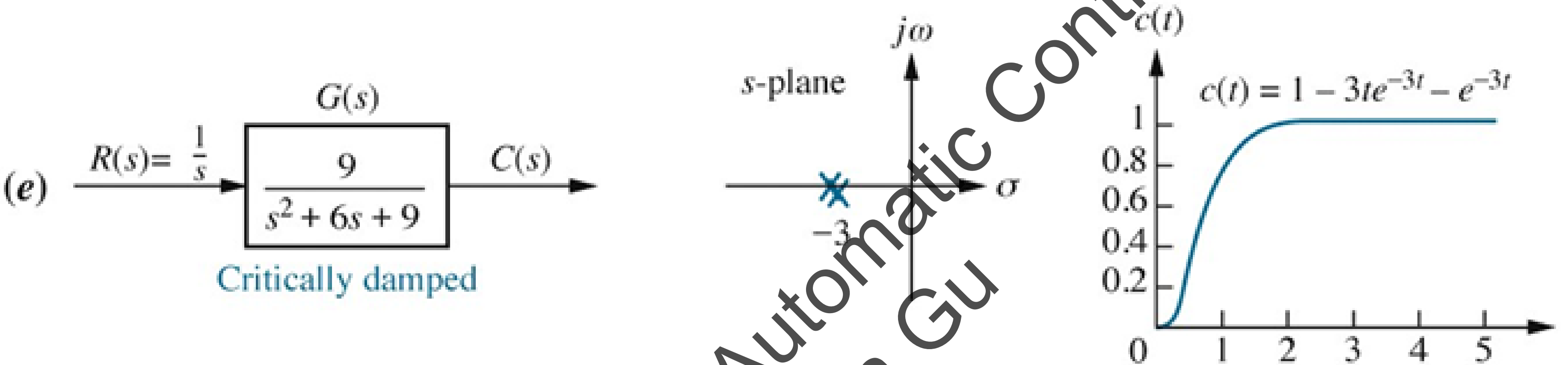
Undamped Response (Two imaginary poles)



$$C(s)=\frac{9}{s\left(s^{2}+9\right)}$$

$$c(t)=K_{1}+K_{2}\cos(3t-\phi)$$

Critically Damped Response (Two repeated real poles)



$$C(s)=\frac{9}{s\left(s^2+6s+9\right)}=\frac{9}{s\left(s+3\right)^2}$$

$$c(t)=K_1+K_2e^{-3t}+K_3te^{-3t}$$

1. *Overdamped responses*

Poles: Two real at $-\sigma_1, \ -\sigma_2$

Natural response: Two exponentials with time constants equal to the reciprocal of the pole locations, or

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 e^{-\sigma_2 t}$$

2. *Underdamped responses*

Poles: Two complex at $-\sigma_d \pm j\omega_d$

Natural response: Damped sinusoid with an exponential envelope whose time constant is equal to the reciprocal of the pole's real part. The radian frequency of the sinusoid, the damped frequency of oscillation, is equal to the imaginary part of the poles, or

$$c(t) = Ae^{-\sigma_d t}\cos(\omega_d t - \phi)$$

3. *Undamped responses*

Poles: Two imaginary at $\pm j\omega_1$

Natural response: Undamped sinusoid with radian frequency equal to the imaginary part of the poles, or

$$c(t) = A\cos(\omega_1 t - \phi)$$

4. *Critically damped responses*

Poles: Two real at $-\sigma_1$

Natural response: One term is an exponential whose time constant is equal to the reciprocal of the pole location. Another term is the product of time, t, and an exponential with time constant equal to the reciprocal of the pole location, or

$$c(t) = K_1 e^{-\sigma_1 t} + K_2 t e^{-\sigma_1 t}$$

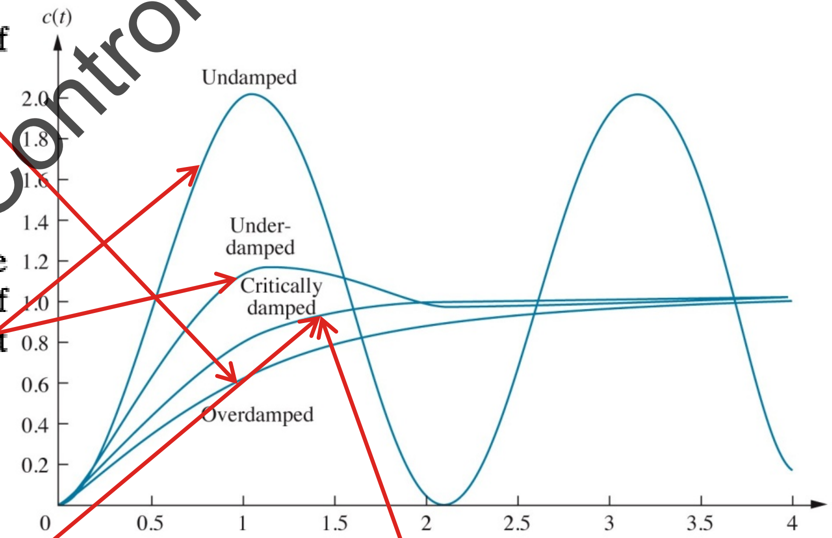


Figure 4.10

The critically damped case is the division between the overdamped cases and the underdamped cases and is the fastest response without overshoot.

Skill-Assessment Exercise 4.3

WileyPLUS WPCS Control Solutions

PROBLEM: For each of the following transfer functions, write, by inspection, the general form of the step response:

a. $G(s) = \dfrac{400}{s^2 + 12s + 400}$

b. $G(s) = \dfrac{900}{s^2 + 90s + 900}$

c. $G(s) = \dfrac{225}{s^2 + 30s + 225}$

d. $G(s) = \dfrac{625}{s^2 + 625}$

ANSWERS:

a. $c(t) = A + Be^{-6t}\cos(19.08t + \phi)$

b. $c(t) = A + Be^{-78.54t} + Ce^{-11.46t}$

c. $c(t) = A + Be^{-15t} + Cte^{-15t}$

d. $c(t) = A + B\cos(25t + \phi)$

The complete solution is located at www.wiley.com/college/nise.

Skill-Assessment Exercise 4.3

WileyPLUS
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a. $G(s) = \dfrac{400}{s^2 + 12s + 400}$

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c. $G(s) = \dfrac{225}{s^2 + 30s + 225}$

d. $G(s) = \dfrac{625}{s^2 + 625}$

4.3

a. Since poles are at $-6 \pm j19.08$, $c(t) = A + Be^{-6t}\cos(19.08t + \phi)$.

b. Since poles are at -78.54 and -11.46, $c(t) = A + Be^{-78.54t} + Ce^{-11.4t}$.

c. Since poles are double on the real axis at -15 $c(t) = A + Be^{-15t} + Cte^{-15t}$.

d. Since poles are at $\pm j25$, $c(t) = A + B\cos(25t + \phi)$.

The General Second-Order System

- Two physically meaningful specifications: natural frequency ω_n and damping ration ζ.
- Natural frequency ω_n: The frequency of oscillation of the system without damping.
- Damping ration ζ: Quantitatively describe this damped oscillation regardless of the time scale. Thus, a system whose transient response goes through three cycles in a millisecond before reaching the steady state would have the same measure as a system that went through three cycles in a millennium before reaching the steady state.

- A viable definition for this quantity is one that compares the exponential decay frequency of the envelope to the natural frequency.

$$\zeta = \frac{\text{Exponential decay frequency}}{\text{Natural frequency}}$$

$$G(s)=\frac{b}{s^2+as+b}$$

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$

(1)

Without damping, poles are imaginary and then *a=0*

$$G(s)=\frac{b}{s^2+b}$$

$$s_{1,2}=\pm j\sqrt{b},K\cos\sqrt{b}t$$

$$\omega_n=\sqrt{b},b=\omega_n^2$$

Real part of the poles is –a/2

$$\zeta=\frac{\text{Exponential decay frequency}}{\text{Natural frequency}}=\frac{|\sigma|}{\omega_n}=\frac{a/2}{\omega_n}$$

$$a=2\zeta\omega_n$$

Example 4.3

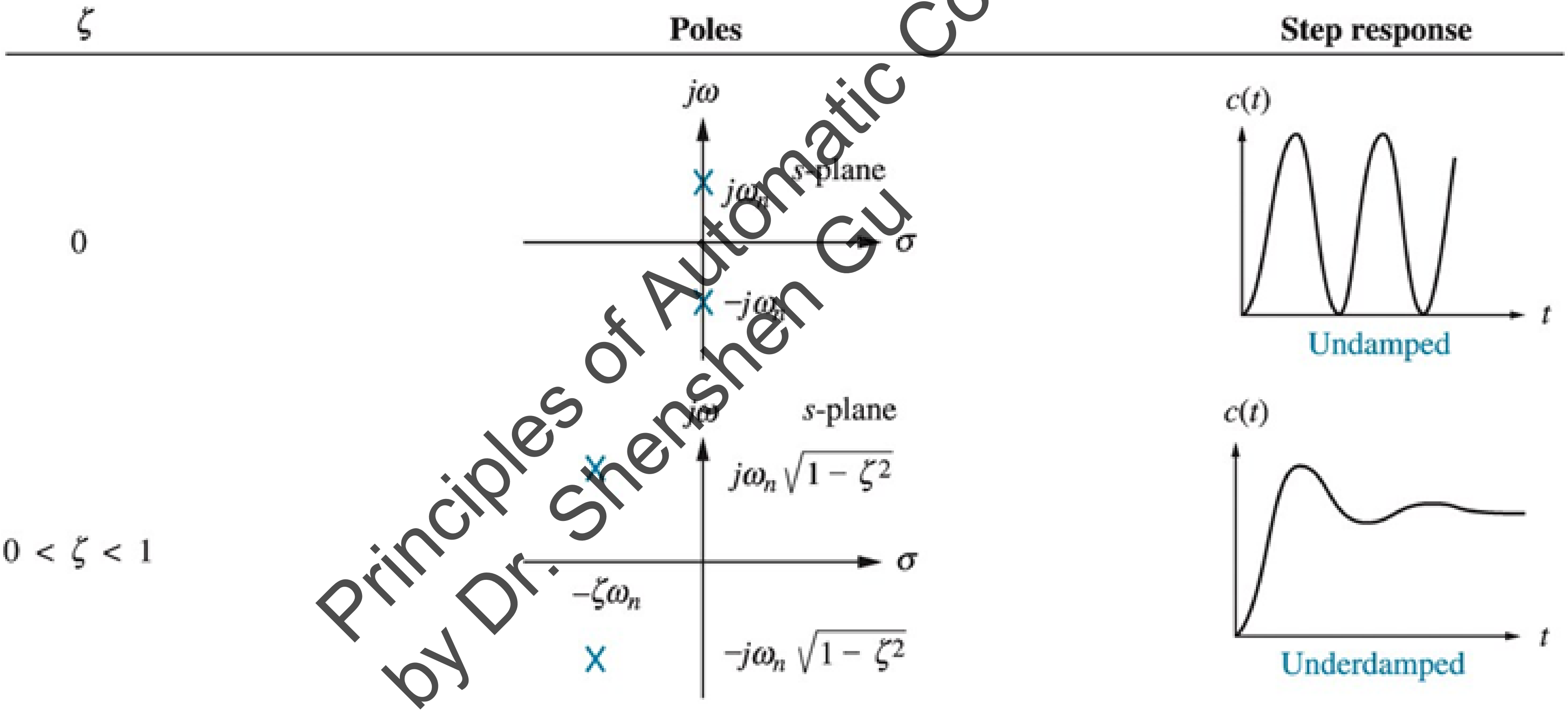
Finding ζ and ω_n For a Second-Order System

PROBLEM: Given the transfer function of Eq. (4.23), find ζ and ω_n.

$$G(s) = \frac{36}{s^2 + 4.2s + 36} \tag{4.23}$$

SOLUTION: Comparing Eq. (4.23) to (4.22), $\omega_n^2 = 36$, from which $\omega_n = 6$. Also, $2\zeta\omega_n = 4.2$. Substituting the value of ω_n, $\zeta = 0.35$.

$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \Rightarrow s_{1,2}=-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$



$$G(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2} \quad \Rightarrow \quad s_{1,2}=-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

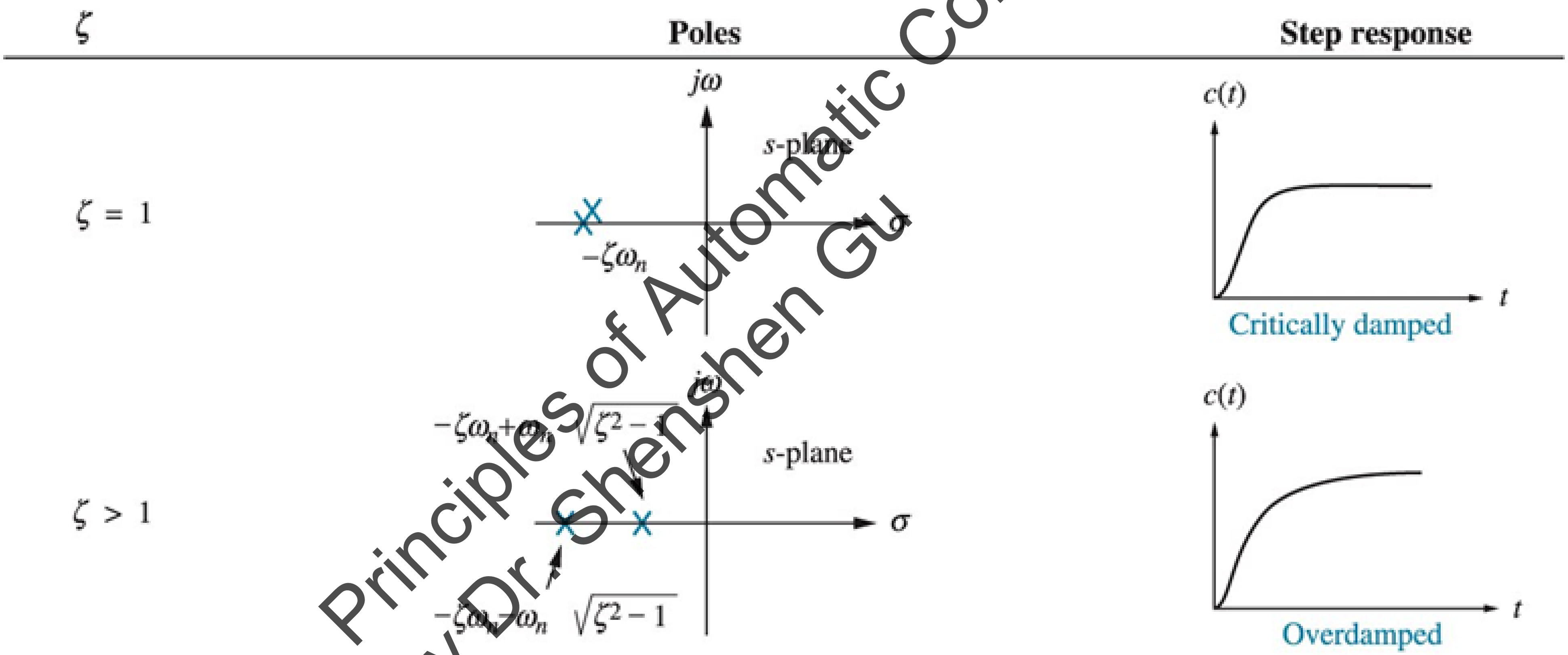


Figure 4.11

Example 4.4

Characterizing Response from the Value of ζ

PROBLEM: For each of the systems shown in Figure 4.12, find the value of ζ and report the kind of response expected.

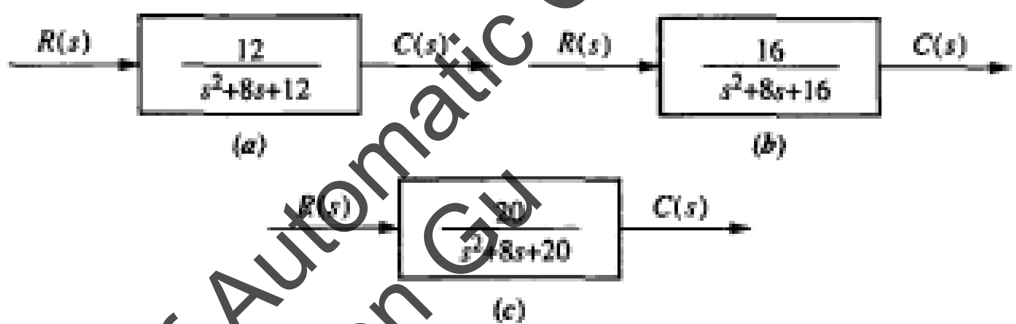


FIGURE 4.12 Systems for Example 4.4

SOLUTION: First match the form of these systems to the forms shown in Eqs. (4.16) and (4.22). Since $a = 2\zeta\omega_n$ and $\omega_n = \sqrt{b}$,

$$\zeta = \frac{a}{2\sqrt{b}} \tag{4.25}$$

Using the values of a and b from each of the systems of Figure 4.12, we find $\zeta = 1.155$ for system (a), which is thus overdamped, since $\zeta > 1$; $\zeta = 1$ for system (b), which is thus critically damped; and $\zeta = 0.894$ for system (c), which is thus underdamped, since $\zeta < 1$.

Skill-Assessment Exercise 4.4

PROBLEM: For each of the transfer functions in Skill-Assessment Exercise 4.3, do the following: (1) Find the values of ζ and ω_n; (2) characterize the nature of the response.

a. $G(s) = \dfrac{400}{s^2 + 12s + 400}$

b. $G(s) = \dfrac{900}{s^2 + 90s + 900}$

c. $G(s) = \dfrac{225}{s^2 + 30s + 225}$

d. $G(s) = \dfrac{625}{s^2 + 625}$

ANSWERS:

a. $\zeta = 0.3$, $\omega_n = 20$; system is underdamped

b. $\zeta = 1.5$, $\omega_n = 30$; system is overdamped

c. $\zeta = 1$, $\omega_n = 15$; system is critically damped

d. $\zeta = 0$, $\omega_n = 25$; system is undamped

The complete solution is located at www.wiley.com/college/nise.

4.4

a. $\omega_n = \sqrt{400} = 20$ and $2\zeta\omega_n = 12$; $\therefore \zeta = 0.3$ and system is underdamped.

b. $\omega_n = \sqrt{900} = 30$ and $2\zeta\omega_n = 90$; $\therefore \zeta = 1.5$ and system is overdamped.

c. $\omega_n = \sqrt{225} = 15$ and $2\zeta\omega_n = 30$; $\therefore \zeta = 1$ and system is critically damped.

d. $\omega_n = \sqrt{625} = 25$ and $2\zeta\omega_n = 0$; $\therefore \zeta = 0$ and system is undamped.

Underdamped Second-Order Systems

- The response of a second order system to a step input is given by:

$$C(s)=\frac{\omega_n^2}{s\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}=\frac{K_1}{s}+\frac{K_2 s+K_3}{s^2+2\zeta\omega_n s+\omega_n^2}$$

- If ζ<1 (the underdamped case)

$$C(s)=\frac{1}{s}-\frac{\left(s+\zeta\omega_n\right)+\frac{\zeta}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{\left(s+\zeta\omega_n\right)^2+\omega_n^2\left(1-\zeta^2\right)}$$

$$c(t)=1-e^{-\zeta\omega_n t}\left(\cos\omega_n\sqrt{1-\zeta^2}t+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}t\right)$$

$$=1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\cos\left(\omega_n\sqrt{1-\zeta^2}t-\phi\right)$$

$$\phi=\tan^{-1}\left(\zeta/\sqrt{1-\zeta^2}\right)$$

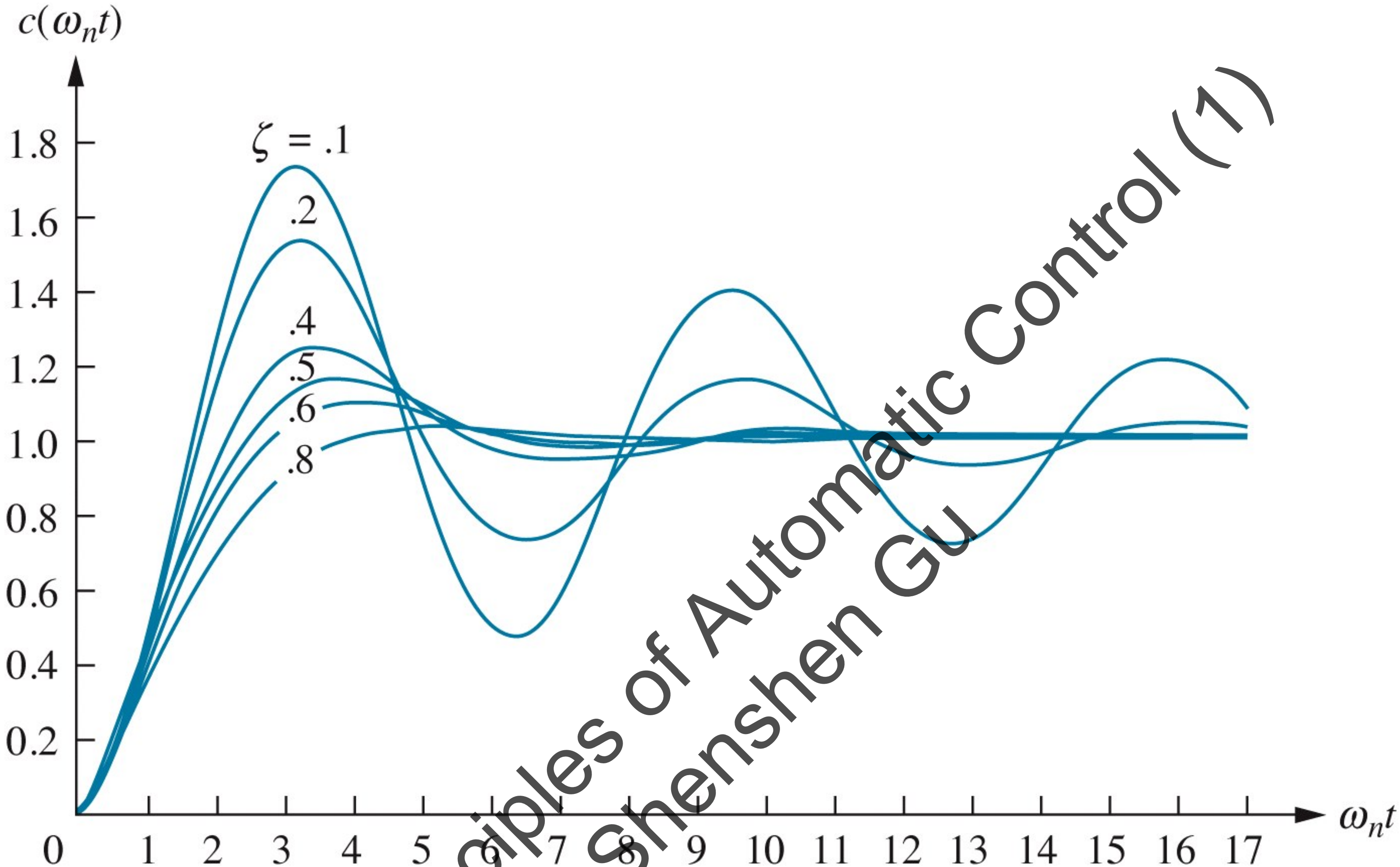


Figure 4.13

- The lower the value of ζ, the more oscillatory the response.

- Other parameters associated with the underdamped response are rise time, peak time, percent overshoot, and settling time.
  - Rise time, T_r. The time required for the waveform to go from 0.1 of the final value to 0.9 of the final value.
  - Peak time, T_p. The time required to reach the first, or maximum, peak.
  - Percent overshoot, *%OS*. The amount that the waveform overshoots the steady-state, or final, value at the peak time. Expressed as a percentage of the steady-state value.
  - Settling time, T_s. The time required for the transient's damped oscillations to reach and stay within $\pm$2% of the steady-state value.

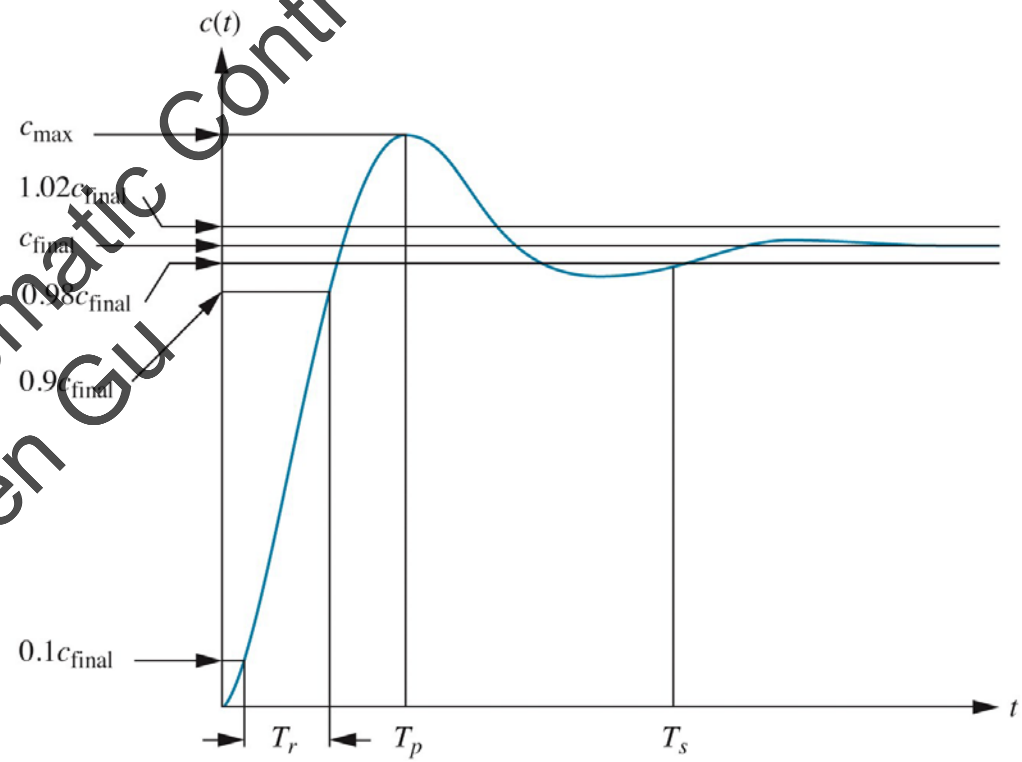


Figure 4.14

- Rise time, peak time, and settling time yield information about the speed of the transient response.

Evaluation of T_p

- T_p is found by differentiating c(t) and finding the first zero crossing after t =0.

$$L\left[\dot{c}(t)\right] = sC(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- Completing squares in the denominator, we have

$$L\left[\dot{c}(t)\right] = \frac{\omega_n^2}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)} = \frac{\frac{\omega_n}{\sqrt{1-\zeta^2}}\omega_n\sqrt{1-\zeta^2}}{(s+\zeta\omega_n)^2 + \omega_n^2(1-\zeta^2)}$$

- Therefore,

$$\dot{c}(t) = \frac{\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin \omega_n \sqrt{1-\zeta^2}\, t$$

- Setting the derivative equal to zero yields $\omega_n\sqrt{1-\zeta^2}\,t = n\pi$
- Or $t = \dfrac{n\pi}{\omega_n\sqrt{1-\zeta^2}}$
- Each value of *n* yields the time for local maxima or minima. Letting *n=0* yields *t=0*, the first point on the curve that has zero slope. The first peak, which occurs at the peak time, T_p, is found by letting *n=1*:

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Evaluation of %OS

$$\%OS = \frac{c_{\max} - c_{final}}{c_{final}} \times 100$$

- The term c_{max} is found by evaluating *c(t)* at the peak time, $c(T_p)$.

$$c_{\max} = c\left(T_p\right) = 1 - e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)}\left(\cos\pi + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\pi\right)$$

$$= 1 + e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)}$$

- For the unit step input, $c_{final}=1$

$$\%OS = e^{-\left(\zeta\pi/\sqrt{1-\zeta^2}\right)} \times 100$$

$$\zeta = \frac{-\ln(\%OS/100)}{\sqrt{\pi^2 + \ln^2(\%OS/100)}}$$

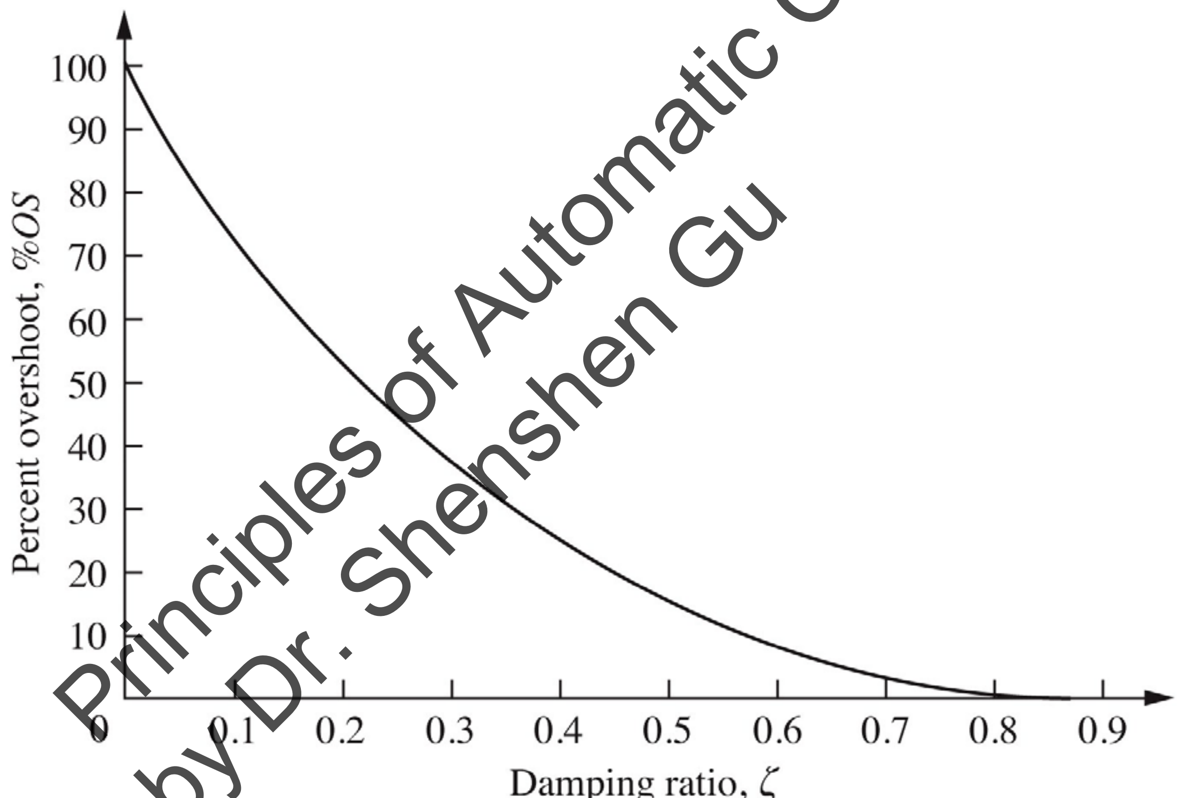


Figure 4.15

Evaluation of T_s

- In order to find the settling time, we must find the time for which output *c(t)* reaches and stays within *2%* of the steady-state value, c_{final}.
- Based on this definition, the settling time is the time it takes for the amplitude of the decaying sinusoid to reach 0.02

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos\left(\omega_n \sqrt{1-\zeta^2}\, t - \phi\right)$$

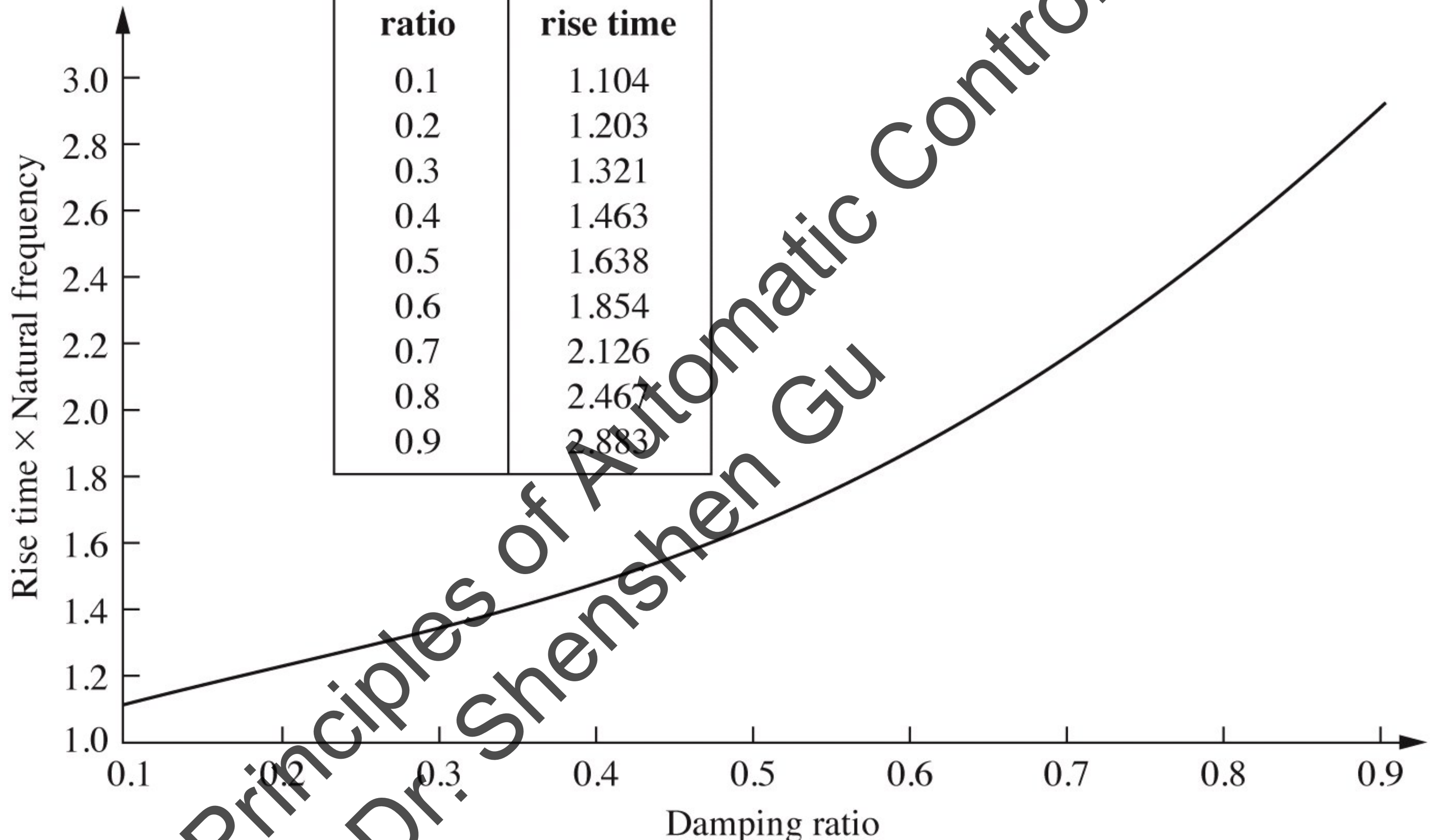
$$\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} = 0.02$$

$$T_s = \frac{-\ln\left(0.02\sqrt{1-\zeta^2}\right)}{\zeta\omega_n}$$

- The numerator varies from 3.91 to 4.74 as ζ varies from 0 to 0.9.

$$T_s = \frac{4}{\zeta\omega_n}$$

Evaluation of T_r



Damping ratio	Normalized rise time
0.1	1.104
0.2	1.203
0.3	1.321
0.4	1.463
0.5	1.638
0.6	1.854
0.7	2.126
0.8	2.467
0.9	2.883

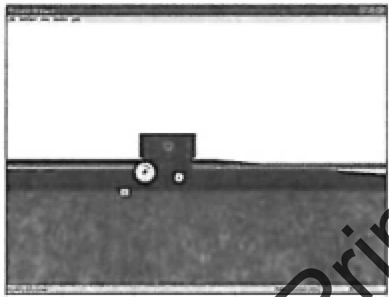
Figure 4.16

Example 4.5

Finding T_p, %OS, T_s, and T_r from a Transfer Function

Virtual Experiment 4.2
Second-Order System Response

Put theory into practice studying the effect that natural frequency and damping ratio have on controlling the speed response of the Quanser Linear Servo in LabVIEW. This concept is applicable to automobile cruise controls or speed controls of subways or trucks.



Virtual experiments are found on WileyPLUS.

PROBLEM: Given the transfer function

$$G(s) = \frac{100}{s^2 + 15s + 100} \tag{4.43}$$

find T_p, $\%OS$, T_s, and T_r.

SOLUTION: ω_n and ζ are calculated as 10 and 0.75, respectively. Now substitute ζ and ω_n into Eqs. (4.34), (4.38), and (4.42) and find, respectively, that $T_p = 0.475$ second, $\%OS = 2.838$, and $T_s = 0.533$ second. Using the table in Figure 4.16, the normalized rise time is approximately 2.3 seconds. Dividing by ω_n yields $T_r = 0.23$ second. This problem demonstrates that we can find T_p, $\%OS$, T_s, and T_r without the tedious task of taking an inverse Laplace transform, plotting the output response, and taking measurements from the plot.

- We now have expressions that relate peak time, percent overshoot, and settling time to the natural frequency and the damping ratio.
- Relate these quantities to the location of the poles that generate these characteristics.
- The radial distance from the origin to the pole is the natural frequency ω_n and the $cos\theta=\zeta$

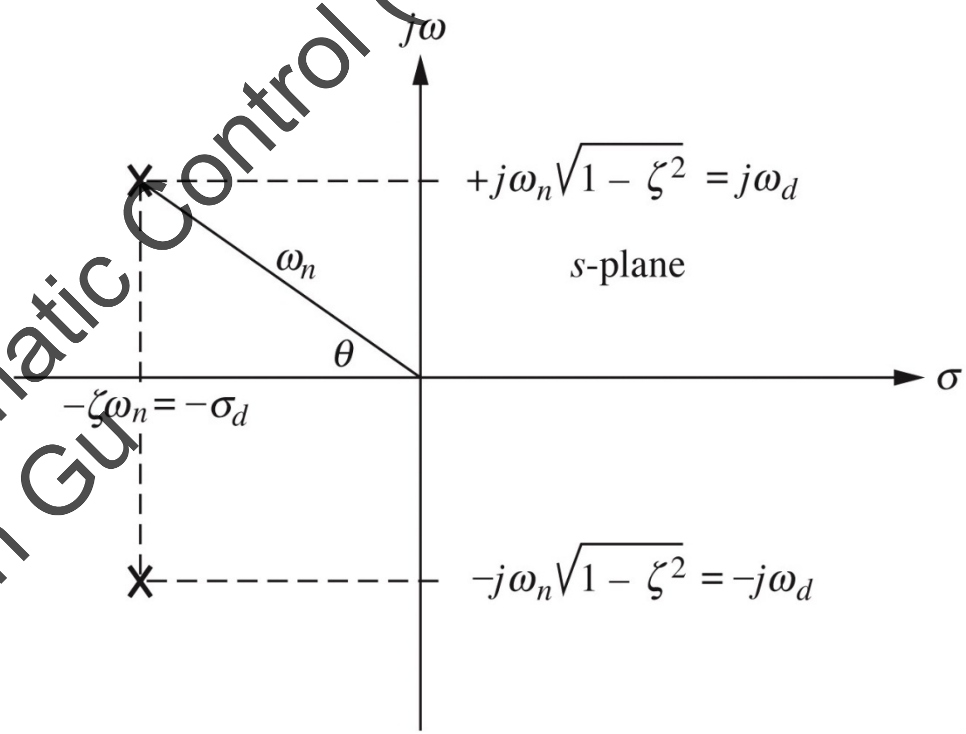


Figure 4.17

- We evaluate peak time and settling time in terms of the pole location.

$$T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_d}$$

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{\sigma_d}$$

- where ω_d is the imaginary part of the pole and is called the damped frequency of oscillation, and σ_d is the magnitude of the real part of the pole and is the exponential damping frequency.

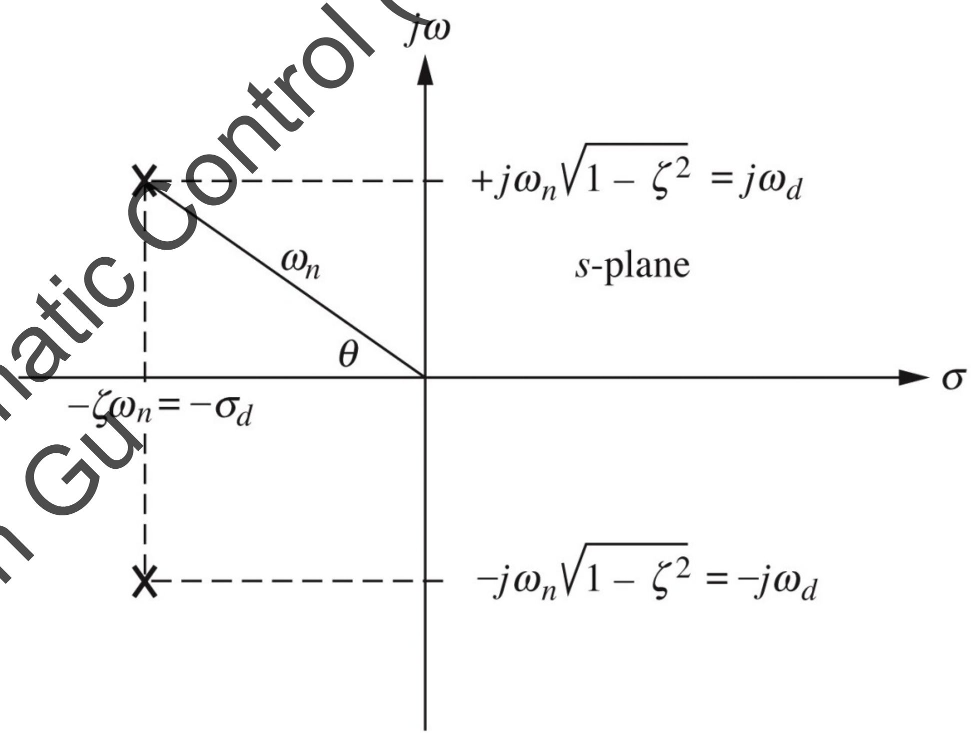


Figure 4.17

- T_p is inversely proportional to the imaginary part of the pole. Since horizontal lines on the s-plane are lines of constant imaginary value, they are also lines of constant peak time.
- Settling time is inversely proportional to the real part of the pole. Since vertical lines on the s-plane are lines of constant real value, they are also lines of constant settling time.
- Since $\zeta = \cos\theta$, radial lines are lines of constant ζ. Since percent overshoot is only a function of ζ, radial lines are thus lines of constant percent overshoot, *%OS*.

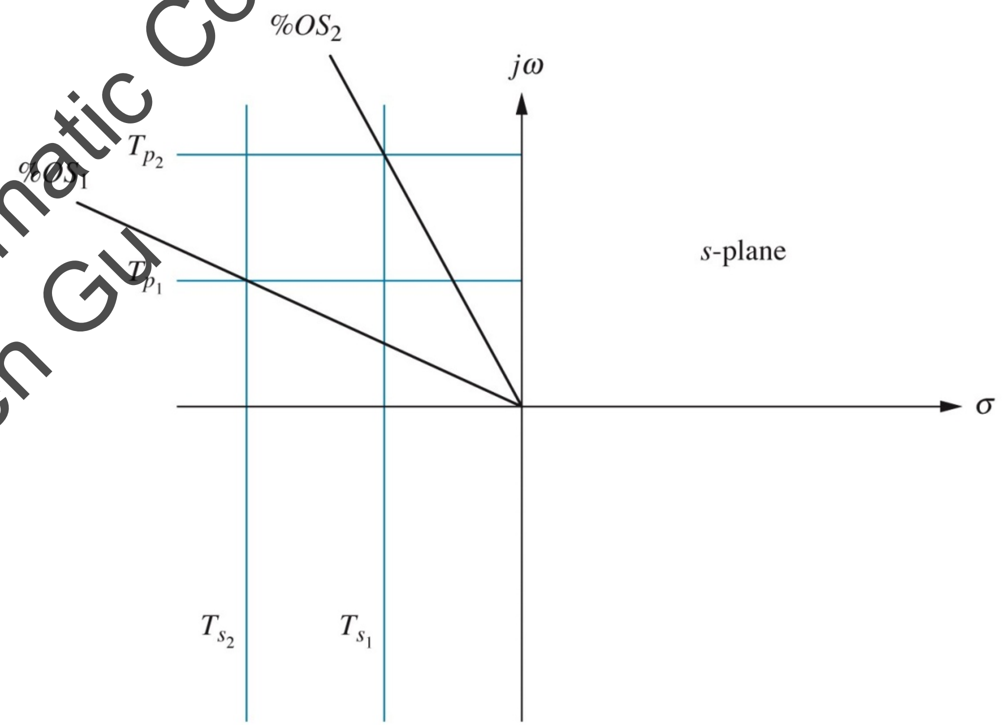
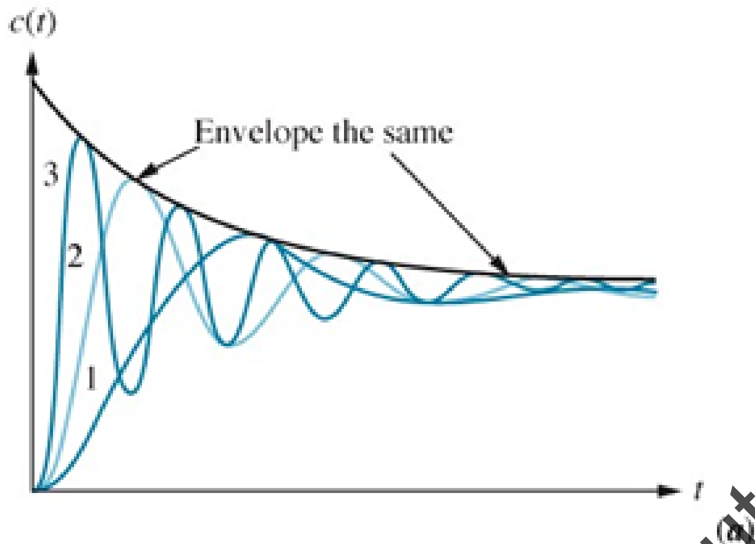
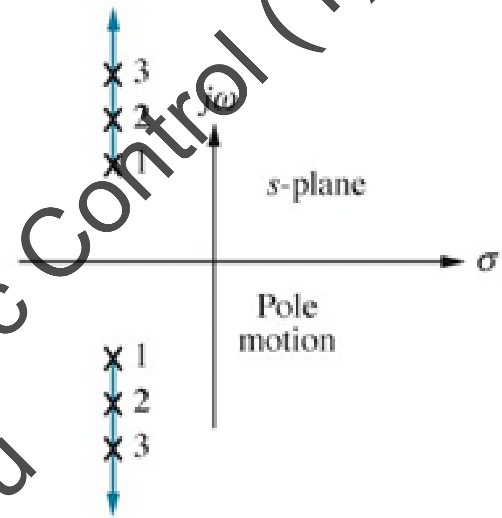


Figure 4.18



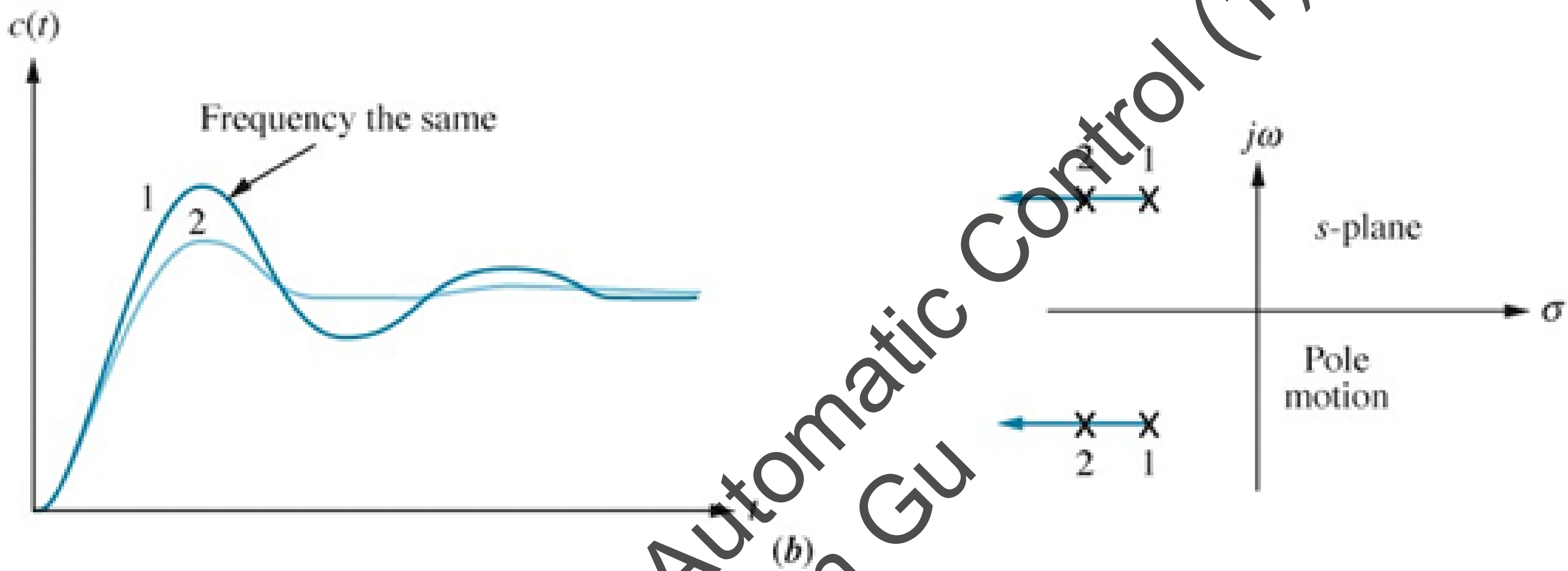


$$\sigma_{d1} = \sigma_{d2} = \sigma_{d3} \Rightarrow T_{s1} = T_{s2} = T_{s3}$$

$$\omega_{d1} < \omega_{d2} < \omega_{d3} \Rightarrow \begin{array}{l} Freq1 < Freq2 < Freq3 \\ T_{p1} > T_{p2} > T_{p3} \end{array}$$

$$\zeta = \cos\theta$$

$$\theta_1 < \theta_2 < \theta_3 \Rightarrow \zeta_1 > \zeta_2 > \zeta_3 \Rightarrow \%OS_1 < \%OS_2 < \%OS_3$$

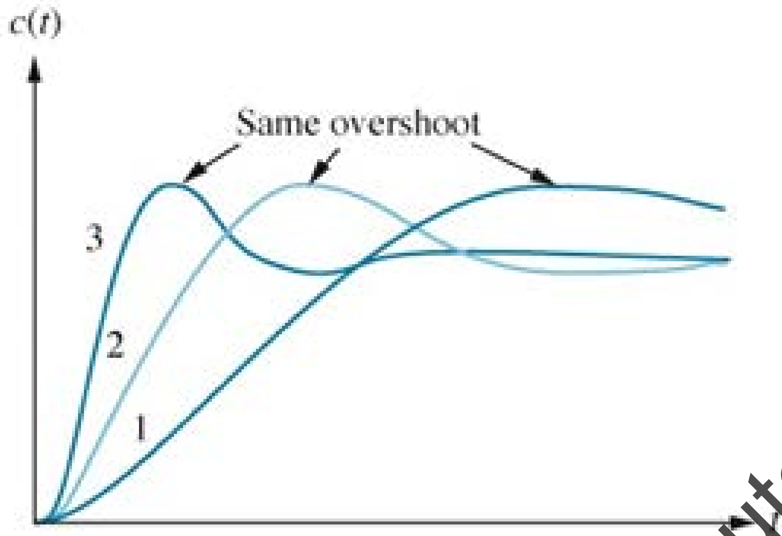


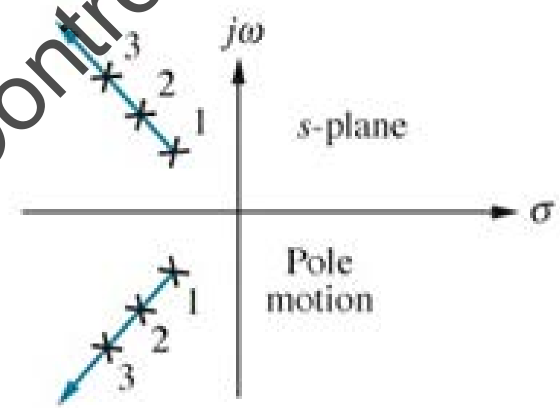
$\omega_{d1} = \omega_{d2}$ → $Freq1 = Freq2$, $T_{p1} = T_{p2}$

$\sigma_{d1} < \sigma_{d2}$ → $T_{s1} > T_{s2}$

$\zeta = \cos\theta$

$\theta_1 > \theta_2$ → $\zeta_1 < \zeta_2$ → $\%OS_1 > \%OS_2$





(c)

$$\zeta = \cos\theta$$

$$\theta_1 = \theta_2 = \theta_3 \Rightarrow \zeta_1 = \zeta_2 = \zeta_3 \Rightarrow \%OS_1 = \%OS_2 = \%OS_3$$

$$\sigma_{d1} < \sigma_{d2} < \sigma_{d3} \Rightarrow T_{s1} > T_{s2} > T_{s3}$$

$$\omega_{d1} < \omega_{d2} < \omega_{d3} \Rightarrow Freq1 < Freq2 < Freq3$$

$$T_{p1} > T_{p2} > T_{p3}$$

Example 4.6

Finding T_p, %OS, and T_s from Pole Location

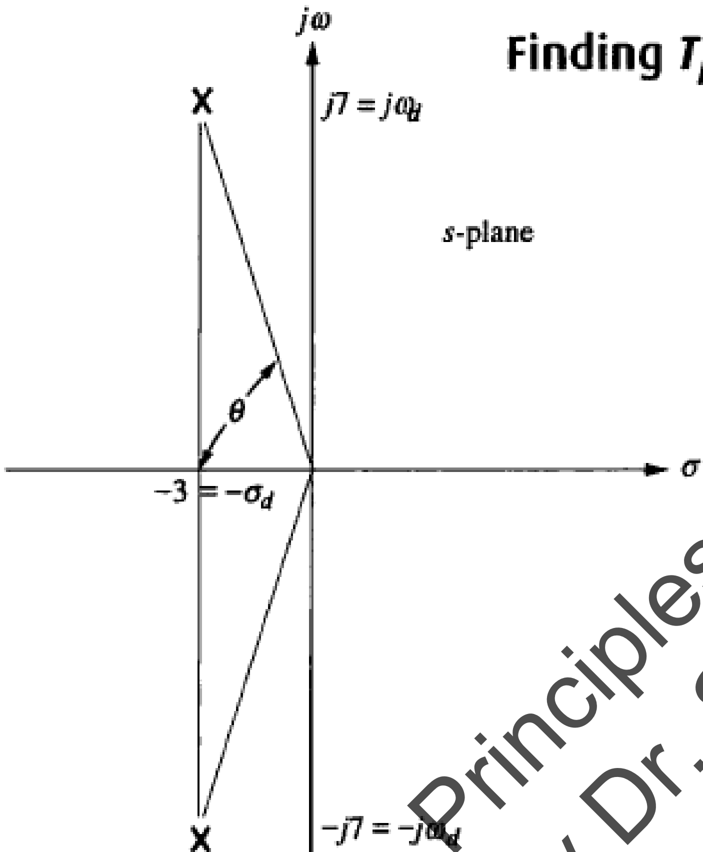


FIGURE 4.20 Pole plot for Example 4.6

PROBLEM: Given the pole plot shown in Figure 4.20, find ζ, ω_n, T_p, $\%OS$, and T_s.

SOLUTION: The damping ratio is given by $\zeta = \cos\theta = \cos[\arctan(7/3)] = 0.394$. The natural frequency, ω_n, is the radial distance from the origin to the pole, or $\omega_n = \sqrt{7^2 + 3^2} = 7.616$. The peak time is

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{7} = 0.449 \text{ second} \tag{4.46}$$

The percent overshoot is

$$\%OS = e^{-(\zeta\pi/\sqrt{1-\zeta^2})} \times 100 = 26\% \tag{4.47}$$

The approximate settling time is

$$T_s = \frac{4}{\sigma_d} = \frac{4}{3} = 1.333 \text{ seconds} \tag{4.48}$$

Skill-Assessment Exercise 4.5

TryIt 4.1

Use the following MATLAB statements to calculate the answers to Skill-Assessment Exercise 4.5. Ellipses mean code continues on next line.

```
numg=361;
deng=[1 16 361];
omegan=sqrt(deng(3)...
 /deng(1))
zeta=(deng(2)/deng(1))...
 /(2*omegan)
Ts=4/(zeta*omegan)
Tp=pi/(omegan*sqrt...
 (1-zeta^2))
pos=100*exp(-zeta*...
 pi/sqrt(1-zeta^2))
Tr=(1.768*zeta^3-...
 0.417*zeta^2+1.039*...
 zeta+1)/omegan
```

WileyPLUS **WPCS** Control Solutions

PROBLEM: Find ζ, ω_n, T_s, T_p, T_r, and $\%OS$ for a system whose transfer function is $G(s) = \dfrac{361}{s^2 + 16s + 361}$.

ANSWERS:

$\zeta = 0.421$, $\omega_n = 19$, $T_s = 0.5$ s, $T_p = 0.182$ s, $T_r = 0.079$ s, and $\%OS = 23.3\%$.

The complete solution is located at www.wiley.com/college/nise.

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 pi/sqrt(1-zeta^2))
Tr=(1.768*zeta^3-...
 0.417*zeta^2+1.039*...
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```

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4.5

$$\omega_n = \sqrt{361} = 19 \text{ and } 2\zeta\omega_n = 16; \quad \therefore \zeta = 0.421.$$

Now, $T_s = \frac{4}{\zeta\omega_n} = 0.5\,s$ and $T_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = 0.182$ s.

From Figure 4.16, $\omega_n T_r = 1.4998$. Therefore, $T_r = 0.079$ s.

Finally, $\%os = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} * 100 = 23.3\%$

Summary

- The step response yields a clear picture of the system's transient response.
- The poles of the input generate the forced response, whereas the system poles generate the transient response.
- We studied two types of systems, first order and second order.
- For first-order systems having a single pole on the real axis, the specification of transient response that we derived was the time constant *1/a*, which is the reciprocal of the real-axis pole location.
- Time constant gives us an indication of the speed of the transient response.

Summary

- A second-order system can exhibit four kinds of behavior: overdamped, underdamped, undamped, and critically damped.
- We found that the value of ζ determines the form of the second-order natural response:
  - If $\zeta = 0$, the response is undamped.
  - If $\zeta < 1$, the response is underdamped.
  - If $\zeta = 1$, the response is critically damped.
  - If $\zeta > 1$, the response is overdamped.
- For the underdamped case we defined several transient response specifications, including these:
  - Percent overshoot, *%OS*
  - Peak time, T_p
  - Settling time, T_s
  - Rise time, T_r